DOING PHYSICS WITH PYTHON

IZHIKEVICH QUADRATIC MODEL FOR SPIKING NEURONS

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Any comments, suggestions or corrections, please email me at

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DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive **GitHub**

mnsIZH02.py

Uses Runge-Kutta method to solve the system differential equation

INTRODUCTION

The model of a neuron described in the book, *Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting* by Izhikevich is not based upon biophysical parameters but is a simple model that faithfully reproduces all the neurocomputational dynamical features of the neuron. The model is a two-dimensional systems having a fast voltage variable v and a slower "recovery" variable u, which may describe activation of the K^+ current or inactivation of the Na^+ current or their combination. The simple model to reproduce spiking and bursting behaviour of many known types of neurons is described by the pair of differential equations

$$C\frac{dv}{dt} = k(v - v_r)(v - v_t) - u + I_{ext}$$

(1)

$$\frac{du}{dt} = a\{b(v - v_r) - u\}$$

$$\text{if} \quad v \geq v_{peak} \quad \Rightarrow \quad v \leftarrow c \quad u \leftarrow u + d$$

This model combines a smooth excitable behaviour leading to the generation of the action potential (spike), and a subsequent discontinuous reset of state variables due to the spike.

Table 1. Model parameters (typical values)

Symbol	Unit / value	Description
t, dt	ms	time, time step
v	mV	membrane potential
dv / dt	mV.ms^{-1}	rate of change of membrane potential
C	pF (pA.ms.mV^{-1})	membrane capacitance $C = 100$ pF
$C\, dv / dt$	pA	capacitor current
v_r	mV	resting membrane potential $v_r \sim -60$ mV
u	pA	recovery variable recovery current The sum of all slow currents that modulate the spike generation mechanism is combined in the phenomenological recovery variable u with outward currents (currents inside to outside) are positive.
du / dt	pA.ms^{-1}	rate of change of recovery variable
I_{ext}	pA	external stimulus current
R	Ω	membrane resistance R = 80 M Ω
τ	ms	membrane time constant C = 100 pF R = 80 M Ω τ = 8 ms
v_{peak}	mV	spike cutoff value $v_{peak} \sim 35$ mV
v_t	mV	instantaneous threshold potential $v_t \sim -40$ mV $v > v_t$ causes the neuron to fire
k	pA.mV^{-1}	constant conductance $k = 1/R$

	$k = 1 / R$	$k \sim 0.7$ pA.mV^{-1}
I_{RB}	pA	rheobase current The minimal amplitude of injected current of infinite duration needed to fire a neuron is called the rheobase. $I_{RB} \sim 50$ pA
a	ms^{-1}	recovery time constant
b	pA.ms^{-1} ($10^{-9}\ \Omega^{-1}$)	constant conductance The sign of *b* determines whether *u* is an amplifying ($b < 0$) or a resonant ($b > 0$) variable. When $b > 0$ the neuron sags in response to hyperpolarized pulses of current; peaks in response to depolarized subthreshold pulses; and produces rebound (post inhibitory) responses.
c	mV	membrane potential reset value Takes into account the action of high-threshold voltage-gated currents activated during the spike, and affect only the after-spike transient behaviour. After spike, membrane hyperpolarize to $c \sim -50$ mV
d	pA	*d* describes the total amount of outward minus inward currents activated during the spike and affecting the after-spike behaviour. $d = 100$ gives a reasonable *f-I* (or *f-S*) relationship in the low-frequency range.

One difficulty in applying the Izhikevich model is the determination of the numerical values of the parameters to be used in stimulations of the time evolution of the membrane potential for different stimuli. The following simulations show how the model can be applied to three types of neurons.

The model described by equation 1 can quantitatively reproduces subthreshold, spiking, and bursting activity of all known types of cortical and thalamic neurons in response to pulses of DC current. The simple model makes testable hypotheses on the dynamic mechanisms of excitability in these neurons and the model is especially suitable for simulations of large-scale models of the brain.

1. Modelling Regular Spiking (RS) Neurons

Regular spiking neurons are the major class of excitatory neurons in the neocortex (part of the cortex of the brain made up of six layers, labelled from the outermost inwards, I to VI). In humans, the neocortex is involved in functions such as sensory perception, generation of motor commands, spatial reasoning and language. RS neurons have a transient K^+ current I_{KT} whose slow inactivation delays the onset of the first spike and increases the inter-spike period, and a persistent K^+ current I_{KP} which is believed to be responsible for the spike frequency adaptation.

When $b < 0$, the depolarizations of v decrease u as if the major slow current is the inactivating K^+ current I_{KT}

The inactivation time constant of I_A is around 30 ms in the subthreshold voltage range $\Rightarrow a \approx 1/30 \approx 0.03$.

Regular spiking neurons fire tonic spikes with adapting (decreasing) frequency in response to injected pulses of DC current. Most of them have Class 1 excitability in the sense that the interspike frequency vanishes when the amplitude of the injected current decreases. These neurons are spiny stellate cells in layer 4 and pyramidal cells in layers 2, 3, 5, and 6. The simulated voltage responses of the model agree quantitatively with the vitro recordings of the layer 5 pyramidal neuron.

Pyramidal neurons (pyramidal cells) are a type of neuron found in areas of the brain including the cerebral cortex, the hippocampus, and the amygdala. Pyramidal neurons are the primary excitation units of the mammalian prefrontal cortex and the corticospinal tract. Key features: triangular shaped soma (after which the neuron is named), a single axon, a large apical dendrite, multiple basal dendrites, and the presence of dendritic spines.

2. Modelling Intrinsically Bursting (IB) Neurons

Intrinsically bursting (IB) neurons generate a burst of spikes at the beginning of a strong depolarizing pulse of current, then switch to tonic spiking mode. They are excitatory pyramidal neurons found in all cortical layers, but are most abundant in layer 5. Some IB neurons respond to the injected pulses of DC current with a burst of high-frequency spikes followed by low-frequency tonic spiking. Many IB neurons burst even when the current is barely super-threshold and not strong enough to elicit a sustained response. However, other IB neurons give a bursting response only to strong current stimuli and weaker stimulation elicits a regular spiking response. In comparison with typical RS neurons, the regular spiking response of IB neurons have a lower firing frequency and higher rheobase (threshold) current, and exhibits shorter latency to the first spike and noticeable afterdepolarizations. The initial high-frequency spiking is caused by the excess of the inward current or the deficit of the outward current needed to repolarize the membrane potential below the threshold. As a result, many spikes are needed to build up outward current to terminate the high-frequency burst. After the neuron recovers, it fires low-frequency tonic spikes because there is a residual outward current (or residual inactivation of inward current) that prevents the occurrence of another burst. Many IB neurons can fire two or more bursts before they switch into tonic spiking mode.

3. Modelling Chattering (CH) Neurons

Chattering (CH) or fast rhythmic bursting (FRB) neurons generate high-frequency repetitive bursts in response to injected depolarizing currents. The magnitude of the DC current determines the inter-burst period, which could be as long as 100 ms or as short as 15 ms and the number of spikes within each burst is typically from two to five. CH neurons are found in visual cortex of adult cats, and morphologically they are spiny stellate or pyramidal neurons of layers 2 - 4, mainly layer 3.

SIMULATIONS

A typical single spike is shown in figure 1. When the membrane potential v reaches the threshold potential vt = - 40 mV, a spike is produced which peaks at v_{peak} = 35 mV. Then a hyperpolarisation resets the membrane potential c = -50 mV.

The shape of the action potential is very similar to the recording of actual neocortical pyramidal neurons but with two discrepancies: recording has a sharper spike upstroke and a smoother spike downstroke. The simple model generates the upstroke of the spike due to the intrinsic (regenerative) properties of the voltage equation. The voltage reset occurs not at the threshold, but at the peak, of the spike. The firing threshold in the simple model is not a parameter, but a property of the bifurcation mechanism of excitability. Depending on the bifurcation of equilibrium, the model may not even have a well-defined threshold, a situation similar to many conductance-based models.

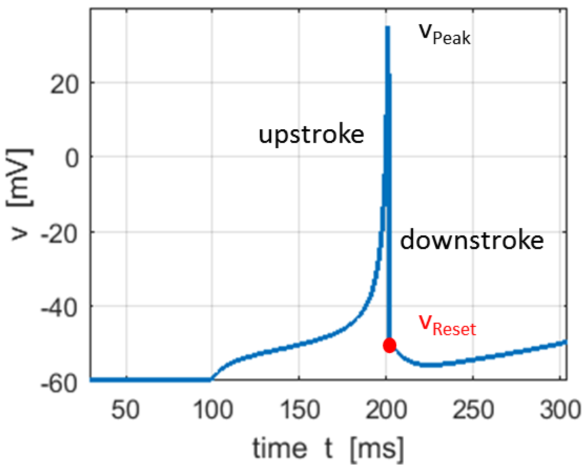


Fig. 1. A single action potential. The slow afterhyperpolarization (AHP) following the reset that is due to the dynamics of the recovery variable u.

For most simulation, a step function of height I_0 is used for the external current stimulus I_{ext} (figure 2). The height of the step stimulus determines the firing rate. If the stimulus current is less than the rheobase current, $I_0 < I_{RB}$ then no spike is produced (figure 3). The external current stimulus is a bifurcation parameter. Slight differences in the bifurcation parameter results in dramatic differences in the response of the system (figure 4). The rheobase current is 51.5 pA. When the external stimulus current is greater than the rheobase current, $I_0 > I_{RB} = 51.5$ pA the neuron spikes as an action potential occurs. When $I_0 > I_{RB}$ the higher the value of the external current stimulus, then the higher the frequency of the spiking (figure 4). Figures 3 and 4 are the membrane potential time evolution plots for an RS neuron.

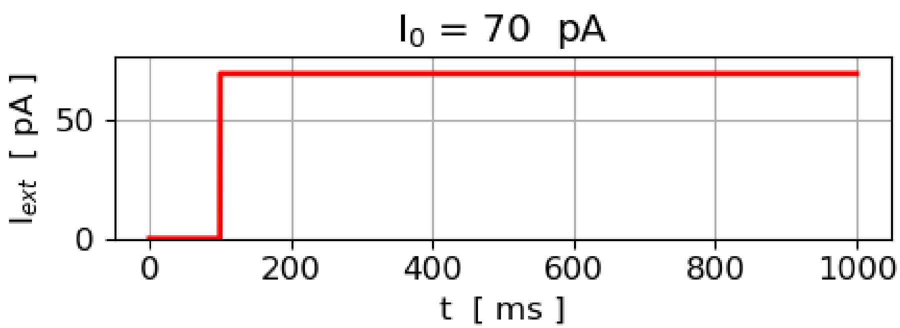


Fig. 2. External current stimulus $I_0 = 70$ pA.

Regular spiking neurons

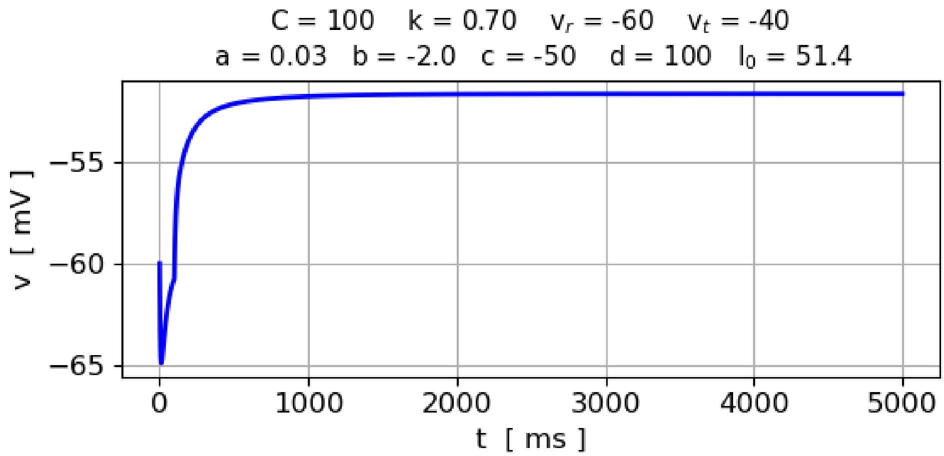


Fig. 3. $I_0 = 51 \text{ pA} < I_{RB}$ No spike is produced.

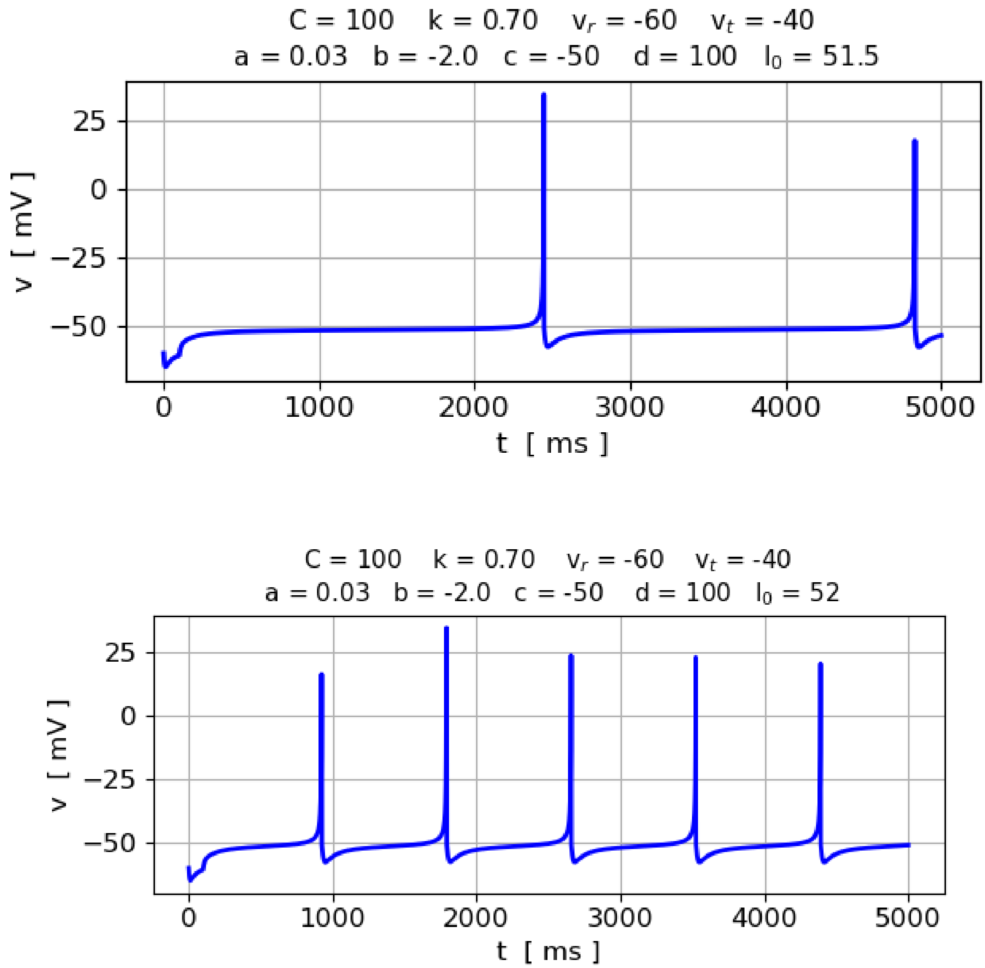


Fig. 4. $I_0 > I_{RB} = 51.5 \text{ pA}$. The greater the value of I_0, the greater the spike frequency.

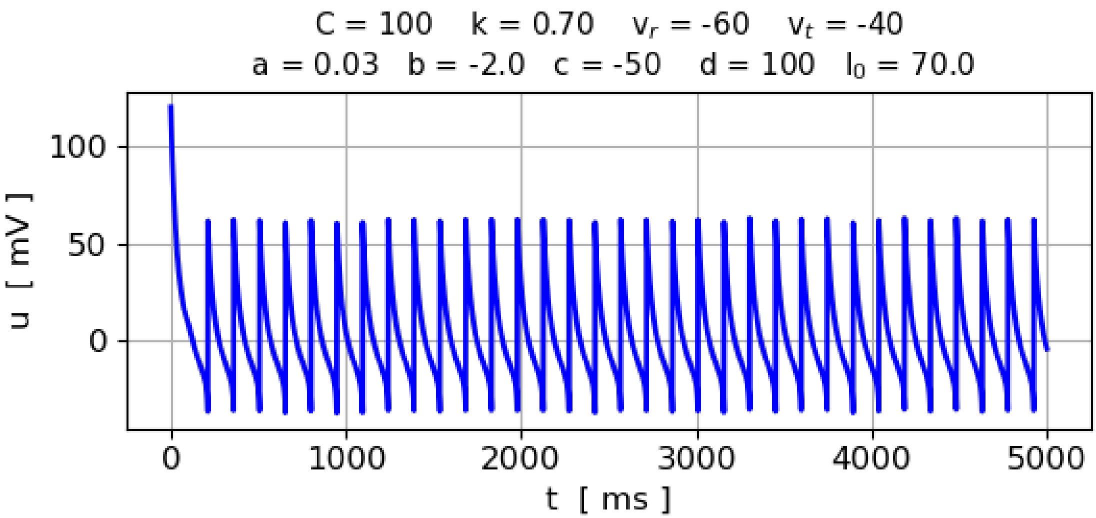


Fig.5. For I_0 = 70 pA there are rapid fluctuations in the recovery variable *u* for a RS neuron.

The inter-spike interval *ISI* and the spike frequency for figures 4 and 5 are

$$I_0 = 51.5\ \text{pA} \quad ISI = 2386\ \text{ms} \quad f = 0.42\ \text{Hz}$$
$$I_0 = 52.0\ \text{pA} \quad ISI = 867\ \text{ms} \quad f = 1.15\ \text{Hz}$$
$$I_0 = 70.0\ \text{pA} \quad ISI = 147\ \text{ms} \quad f = 6.79\ \text{Hz}$$

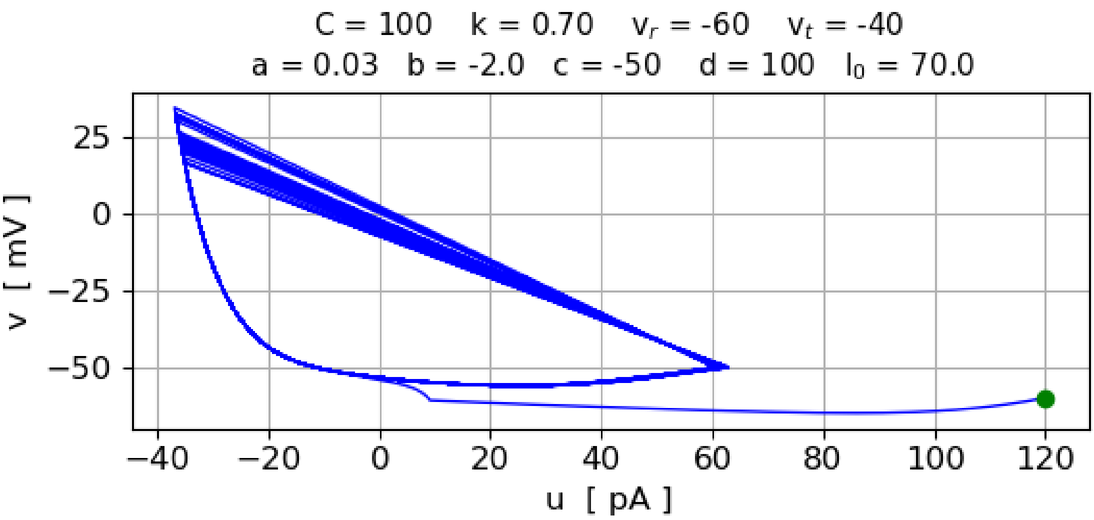


Fig. 6. Phase plot for a RS neuron. After the initial transience, the trajectory settles down to a limit cycle.

Consider the external current to be a ramp stimulating a RS neuron (figure 7). Figure 8 shows the membrane potential response. The neuron does not spike until the stimulus current exceeds the rheobase current. After spiking, the inter-spike interval decreases (frequency of spikes increase) with increasing stimulus (figure 9).

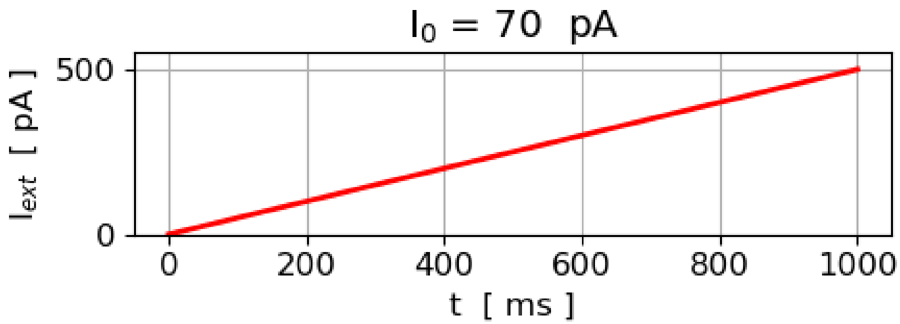


Fig. 7. A ramp external current stimulus.

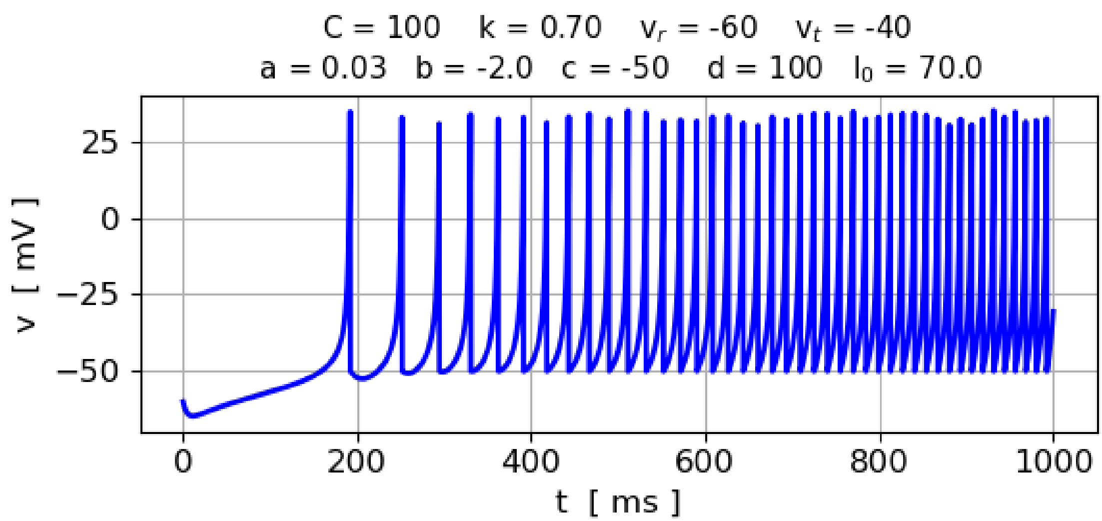


Fig. 8. Membrane potential for a RS neuron stimulated by the ramp current input.

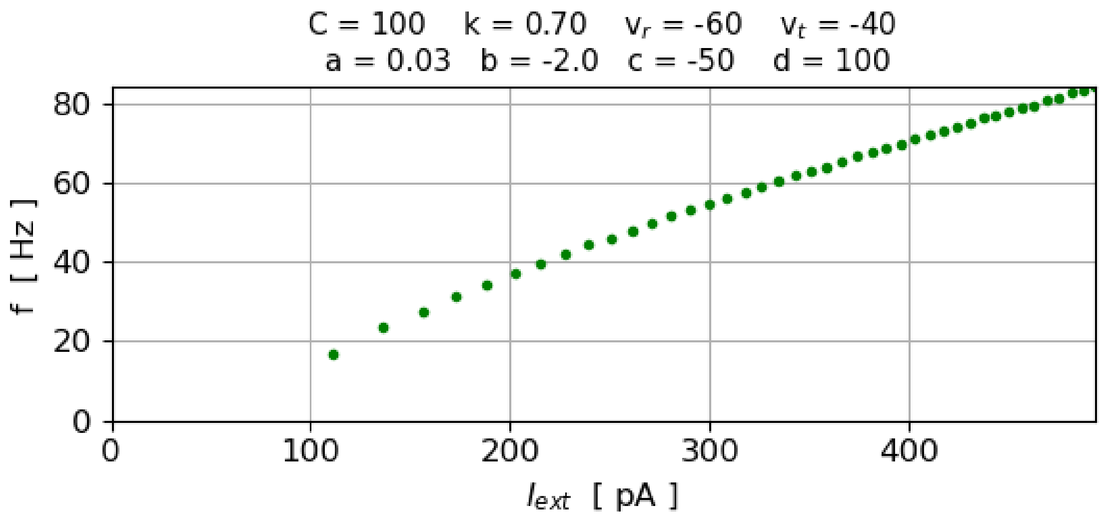
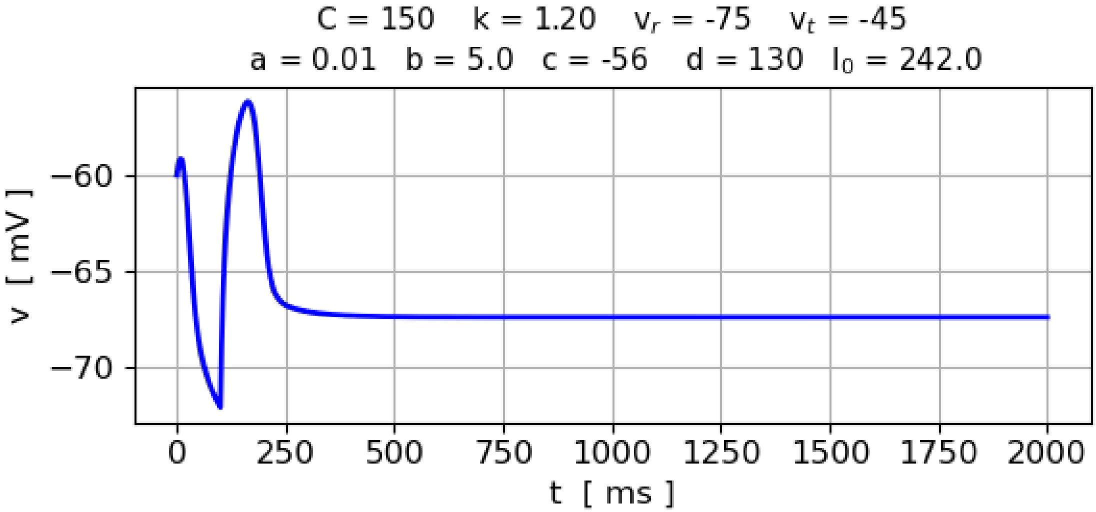


Fig. 9. f-I_{ext} curve: Spike frequency increases as stimulus strength I_{ext} increases.

Intrinsically Bursting (IB) Neurons

The external current stimulus I_{ext} is a bifurcation parameter. Small differences in I_{ext} lead to very different system responses (figure 10). The rheobase current I_{RB} ~ 347 pA. When 243 pA < I_0 < 342 pA a single action potential is generated. When 343 pA < I_0 < 347 a pair of closely spaced action potentials are generated. When I_0 > 347 pA low frequency tonic spiking with adaptation occurs after the single double action potential and the frequency of firing continues to increases with the current stimulus.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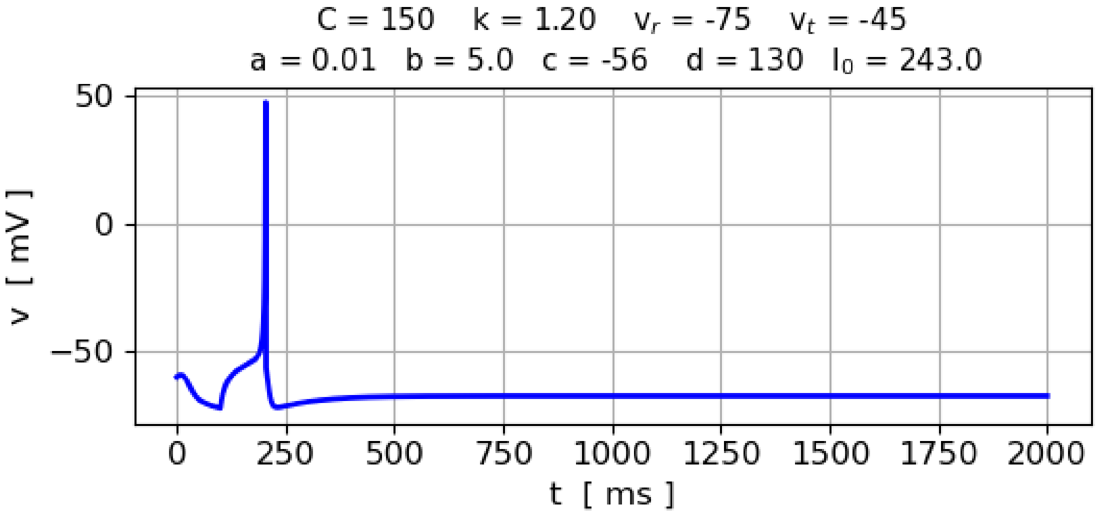
t [ms]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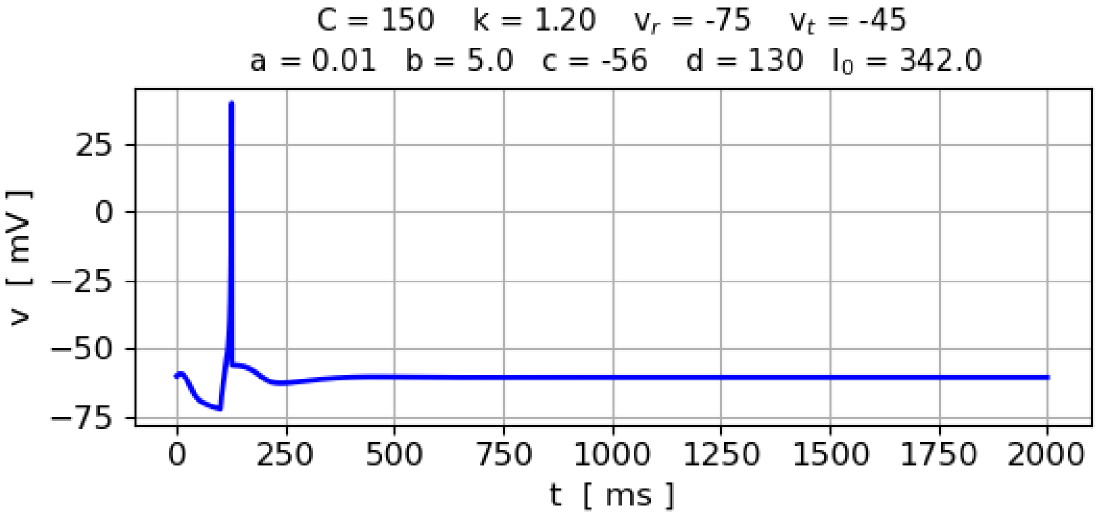
t [ms]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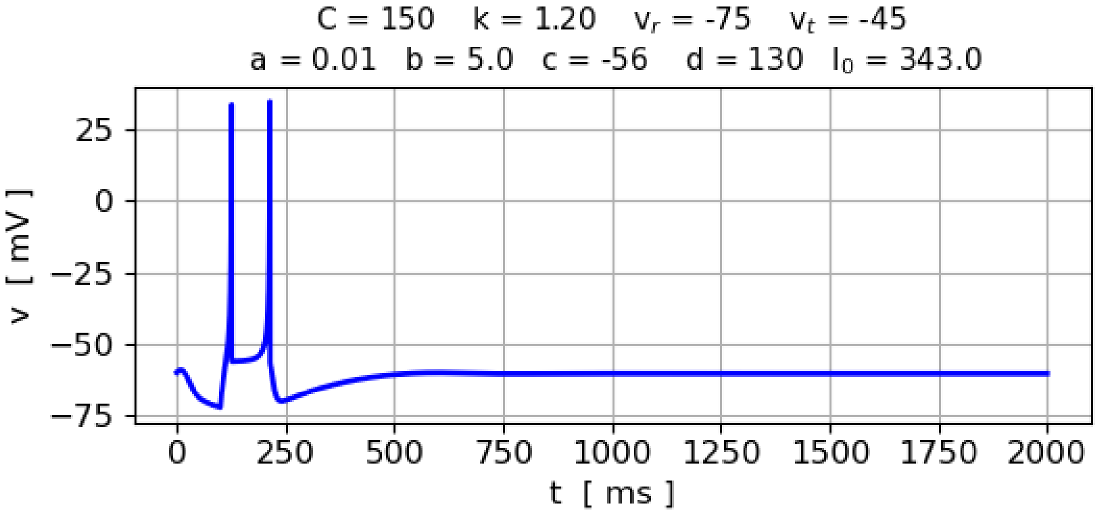
−75
0 250 500 750 1000 1250 1500 1750 2000
t [ms]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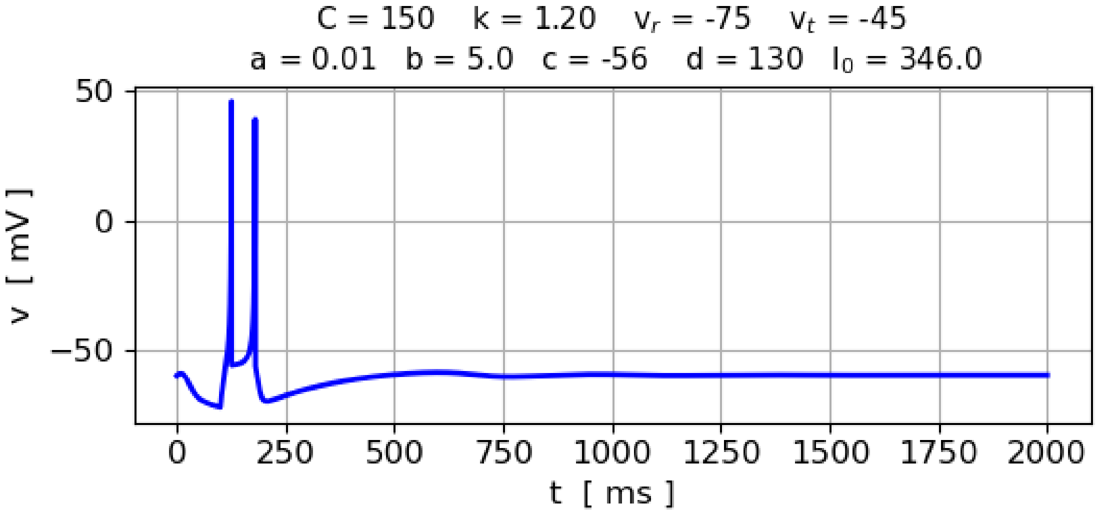
1750
2000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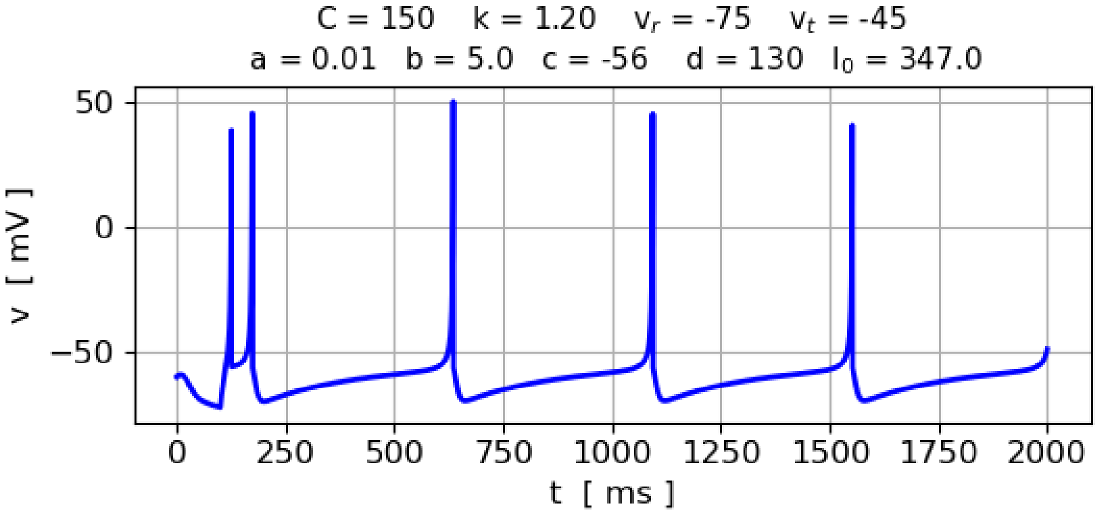
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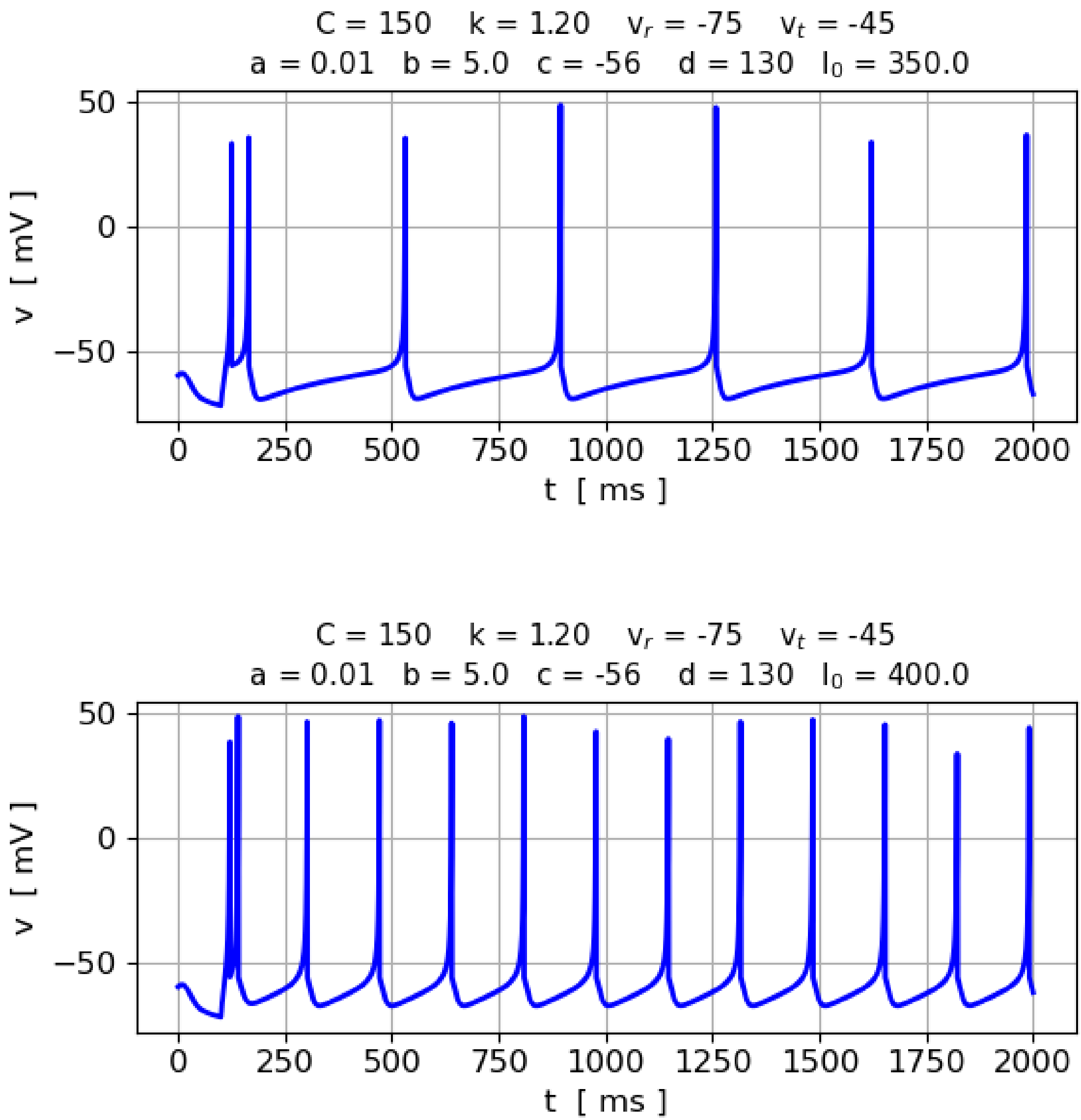


Fig. 10. Time evolution of the membrane potential as a function of the bifurcation parameter I_{ext}. When a spike train is produced the first firing is a double pulse.

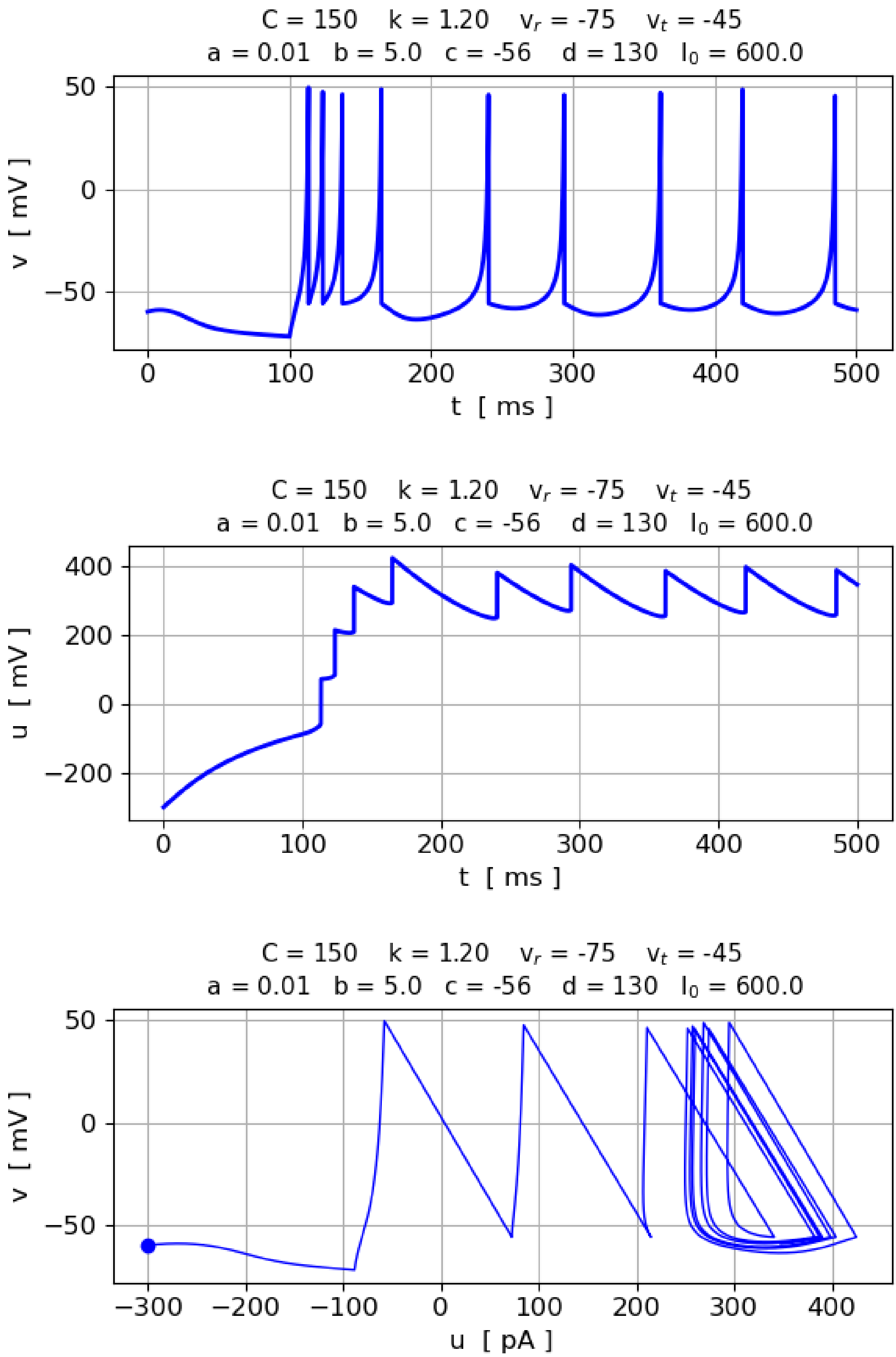
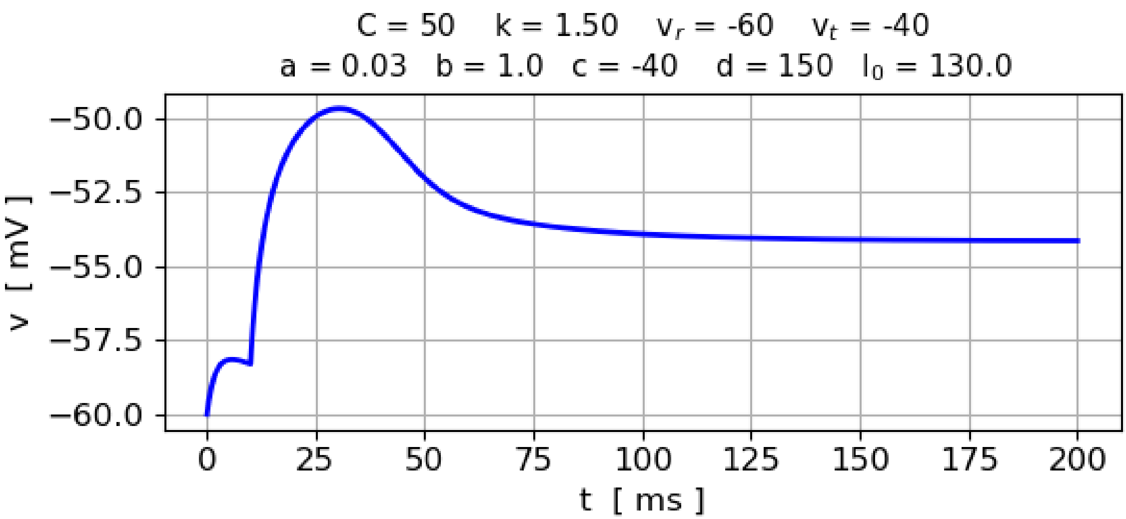
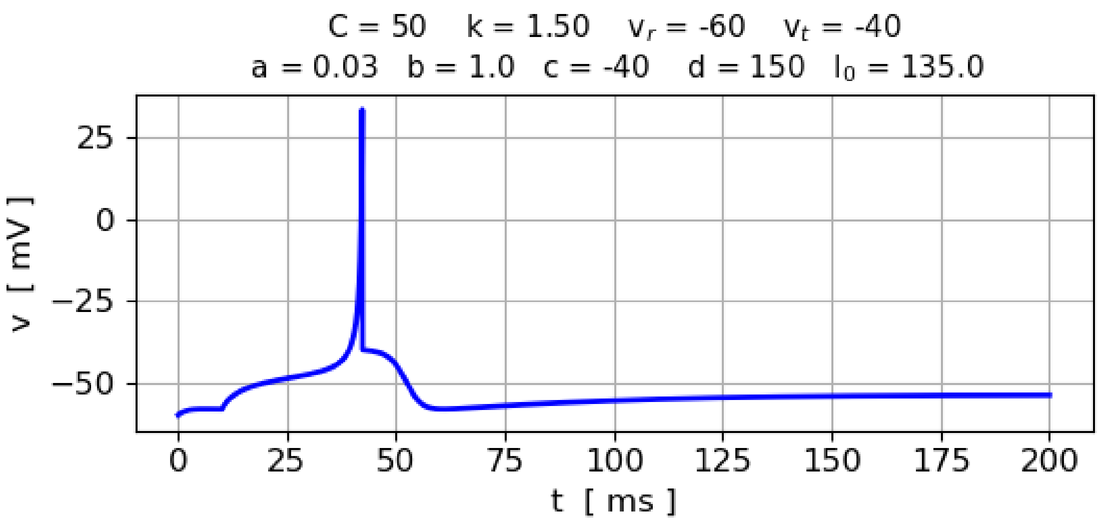


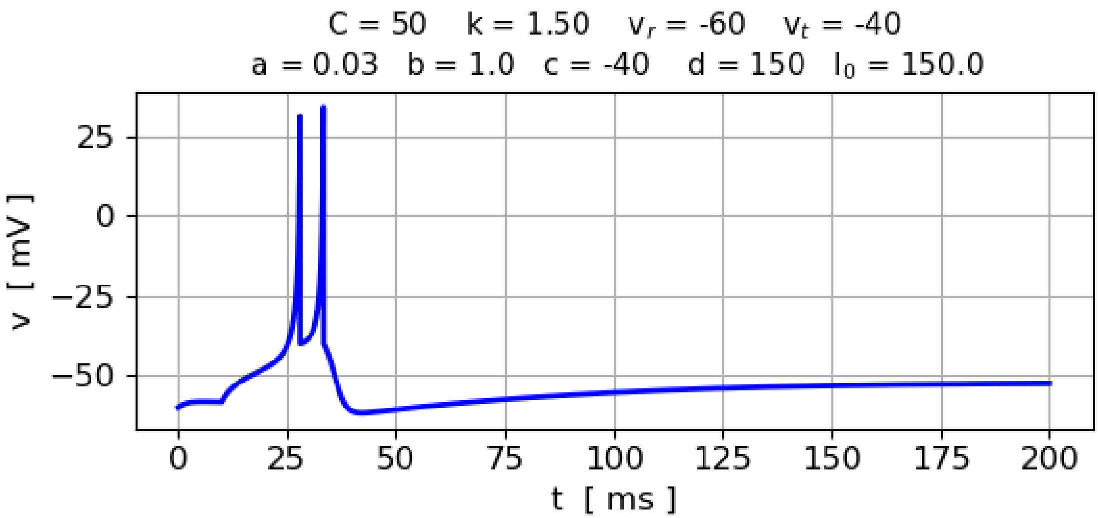
Fig. 11. For a large stimulus I_0 = 600 pA a triple pulse occurs at the start of the spike train.

Chattering (CH) Neurons

The plots of the simulations for CH neurons shown below are very similar to vivo recordings from cat primary visual cortex. Again, the time evolution of the membrane potential is very dependent upon the strength of external current stimulus.



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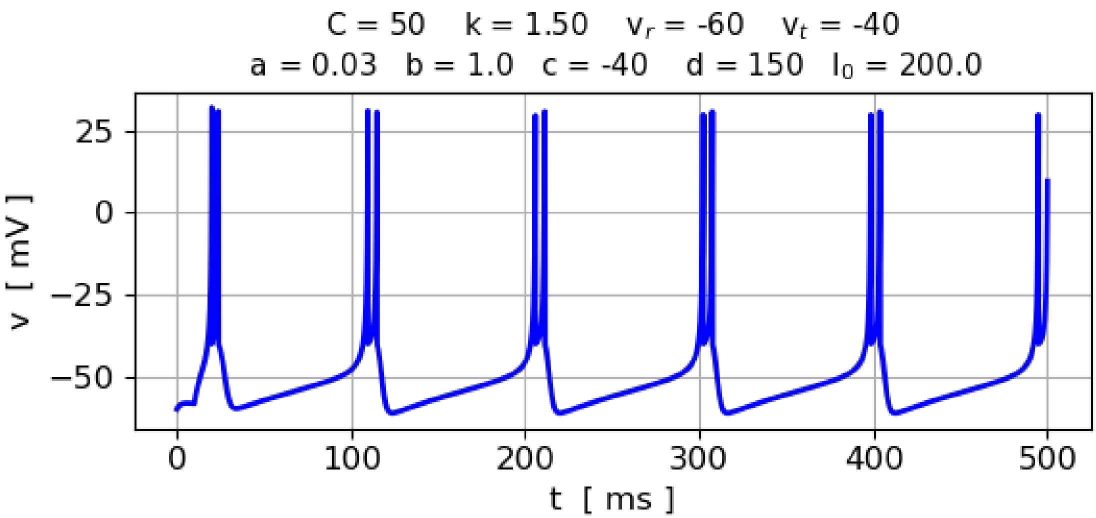
−50
0 25 50 75 100 125 150 175 200
t [ms]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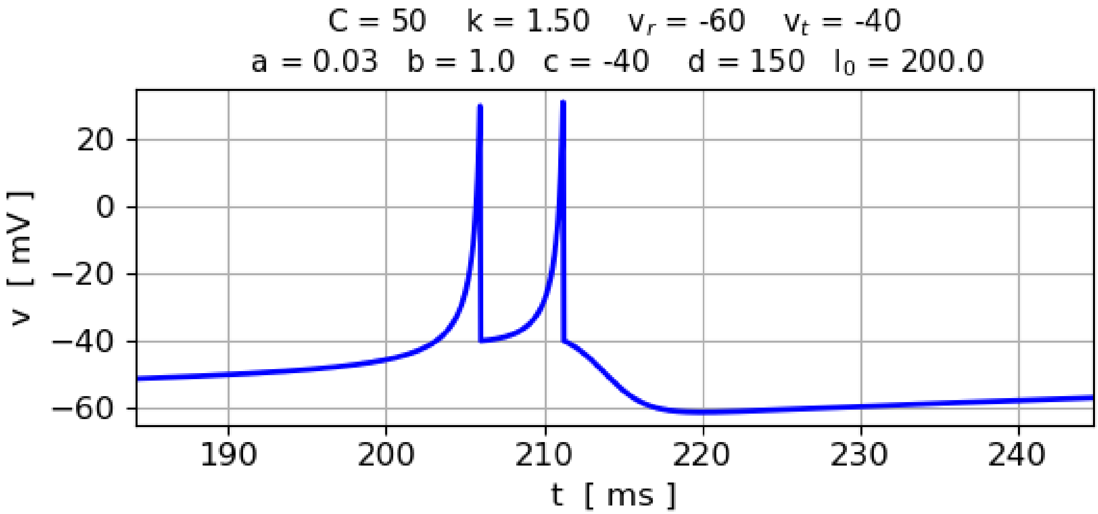
−50
0 100 200 300 400 500
t [ms]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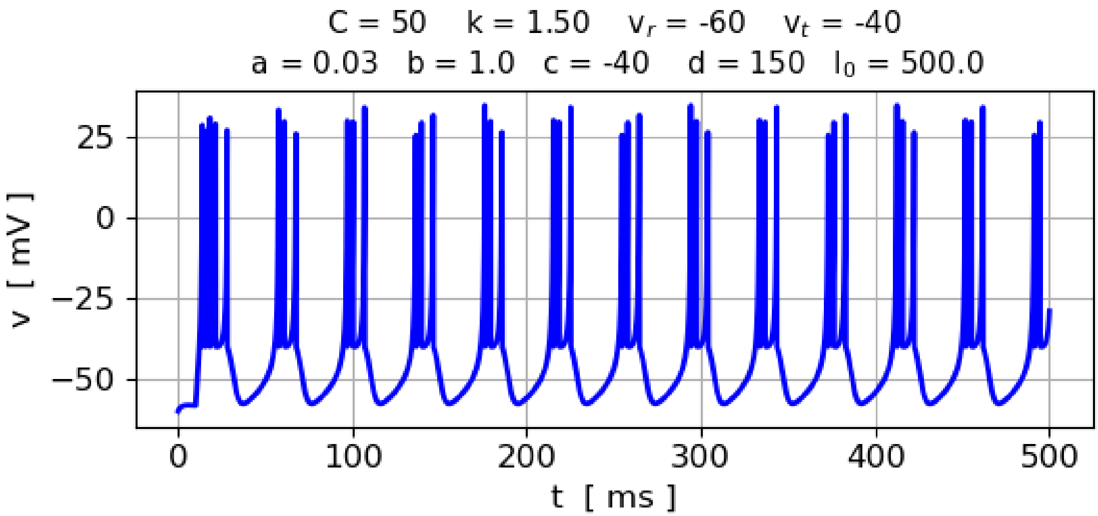
t [ms]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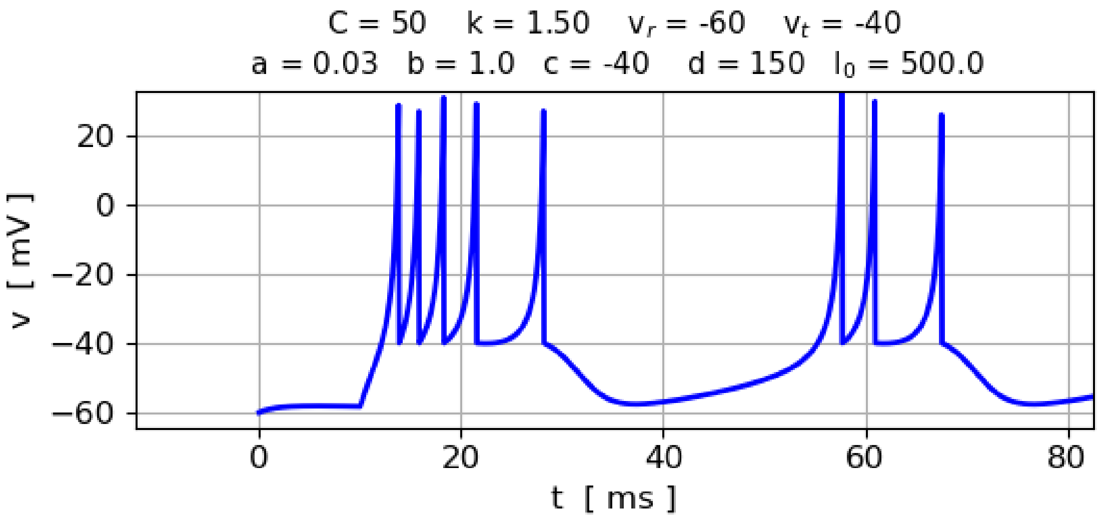
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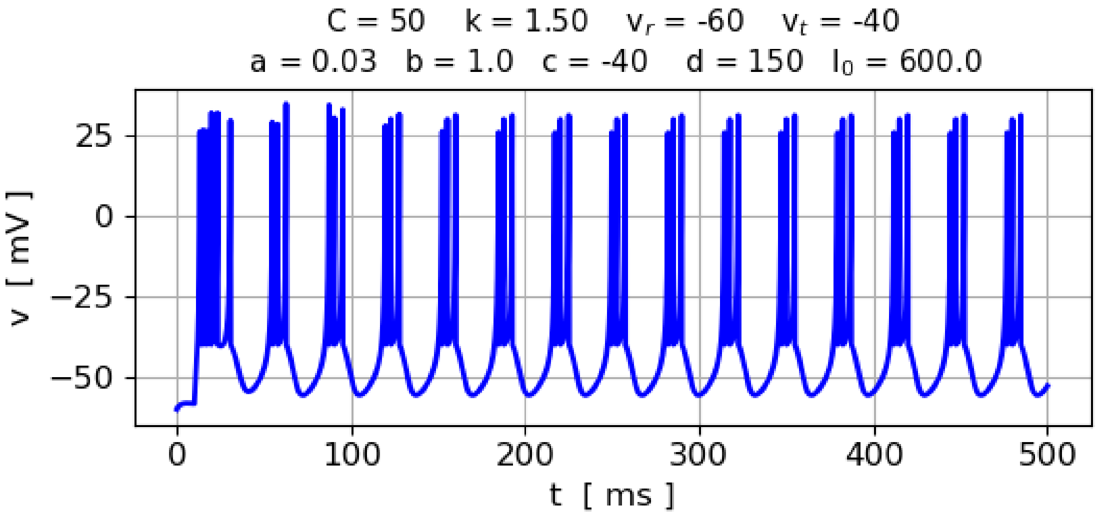
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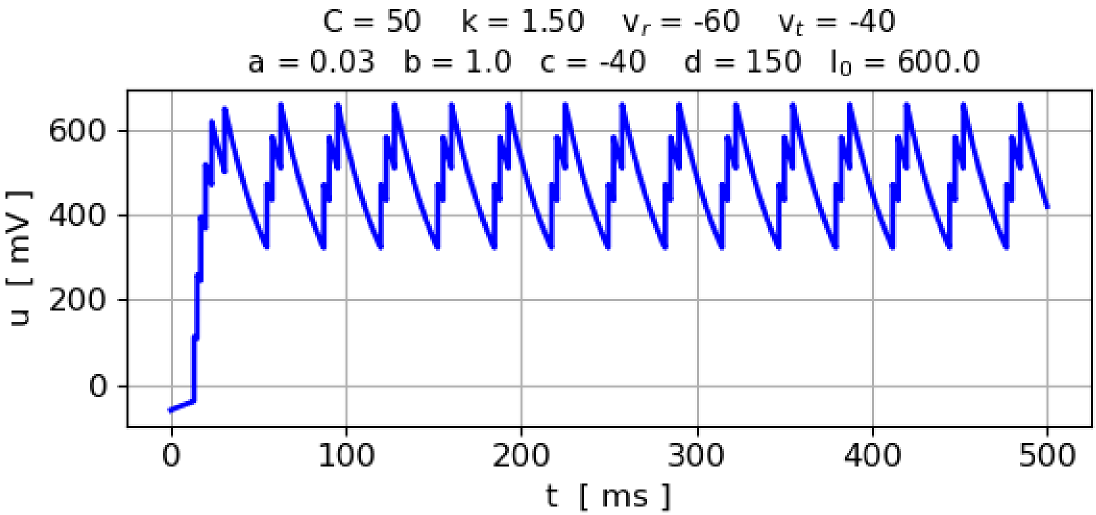
t [ms]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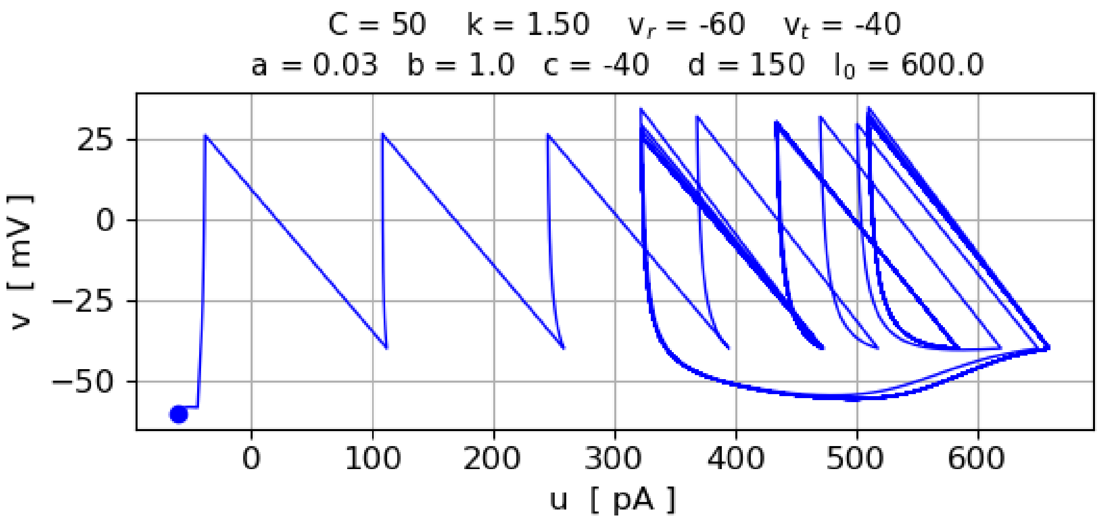
t [ms]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v [mV]
u [pA]

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This article and the Scripts are based upon the papers and book by E. M. Izhikevich

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