DOING PHYSICS WITH MATLAB

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A COMPUTATIONAL APPROACH TO ELECTROMAGNETIC THEORY

CHAPTER 3
INTEGRAL CALCULUS
INTEGRATION

DOWNLOAD DIRECTORIES FOR MATLAB SCRIPTS

Google drive

GitHub

SCRIPTS

simpson1d.m   simpson2d.m

cemCh3.m   cemStokes.m

Reference

David J Griffiths *Introduction to Electrodynamics* (3rd Edition)

Integration is defined for continuous functions. Thus, if a function is continuous in a region, then the integral of $f(x)$ is defined as another function $F(x)$ such that

$$F = \int f(x)\,dx$$

The function $F(x)$ satisfies the condition

$$f(x) = \frac{dF(x)}{dx}$$

SYMBOLIC INTEGRATION

The symbolic function **int** can be used to evaluate both indefinite and definite integrals as shown in the Script **cemCh3.m (CELL 1)**

```
%% Symbolic Integration
close all; clc; clear

% Indefintite integral [1D]
syms x
% Input functiuon >>>>>
  f = sin(2*pi*x)
% Integral
  F = int(f,x)    → -cos(2*pi*x)/(2*pi)
% Definite integral [1D]
  f = exp(-x)*sin(x)/x
% Integral
  F = int(f,x,0,inf)    → Pi/4
% Indefintite integral [1D]
syms x a
% Input functiuon >>>>>
  f = 1/(a^2 + x^2)
% Integral
  F = int(f,x)    → F = atan(x/a)/a
% Definite integral
  f = x*exp(-x)
% Integral
  f = int(f,x,0, inf) →  1
```

NUMERICAL INTEGRATION

Using Matlab makes it is easy to integrate a function numerically. For [1D] integrals we will only use the 1/3 Simpson rule with the function

simpson1d.m

```
function integral = simpson1d(f,a,b)

% [1D] integration - Simpson's 1/3 rule
%       f function     a = lower bound     b = upper bound
%       Must have odd number of data points
%       Simpson's coefficients   1 4 2 4 ... 2 4 1

numS = length(f);          % number of data points
if mod(numS,2) == 1
sc = 2*ones(numS,1);
sc(2:2:numS-1) = 4;
sc(1) = 1; sc(numS) = 1;
h = (b-a)/(numS-1);
integral = (h/3) * f * sc;
else

integral = 'Length of function must be an ODD number'
end
```

Example 1 cemCh3.m (CELL 2)

```
%% [1] integration using Simpson's rule
   clear; close all; clc
% Input: number of grid point N (odd number),  lower limit
a, upper limit b
  N = 999;
  a = 0;  b = pi/2;

  x = linspace(a,b,N);
% Input function >>>>>
  f = sin(x);

% Evaluate integral
  F = simpson1d(f,a,b);
% Output to Commmand Window
  fprintf('integral  F = %2.10e  \n',F)
```

→ integral F = 1.0000000000e+00

Example 2 cemCh3.m (CELL 2)

```
% Input: number of grid point N (odd number),  lower limit
a, upper limit b
  N = 999;
  a = 0;  b = 1;

  x = linspace(a,b,N);
% Input function >>>>>
  q = 0.3; r = 0.9; s = 6;
  f = 1./((x-q).^2+0.01) + 1./((x-r).^2 + 0.04) - s;

% Evaluate integral
  F = simpson1d(f,a,b);
% Output to Commmand Window
  fprintf('integral  F = %2.10e  \n',F)
```

→ integral F = 2.9858325396e+01

Exact value F = 2.985832539549867e+011 *excellent agreement*

[2D] integrals

We can compute the value of double (area or surface) integrals of the form

$$F = \int_{xMin}^{xMax} \int_{yMin}^{yMax} f(x,y)\, dx\, dy$$

You can use the Matlab function **integral2** or the Script **simpson2d.m**

How to use either function is illustrated in the following examples using the Script **cemCh3.m**.

Example 3 integral2 $f(x,y) = 1 + 2x^2 + y^2$ **cemCh3.m (CELL 3)**

```
%% [2D] integration:  Matlab function  integral2
  clear; close all; clc
% Limits   >>>>>
  N = 999;
  xMin = -3; xMax= 3;
  yMin= -4; yMax = 4;
% Grid
 % x = linspace(xMin,xMax, N);
 % y = linspace(yMin,yMax, N);

% Compute integral
  funct = @(x,y) 1 + 2.*x.^2 + y.^2;
  F = integral2(funct,xMin,xMax,yMin,yMax);

  % Output to Commmand Window
  fprintf('integral  F = %2.10e  \n',F)
```

→ integral F = 5.9200000000e+02 (exact 592)

Example 4 simpson2d.m $f(x) = 1 + 2x^2 + y^2$ **cemCh3.m (CELL 4)**

```
%% [2D] integration:  simpson2d.m

  clear; close all; clc
% Limits   >>>>>
  N = 999;
  xMin = -3; xMax= 3;
  yMin= -4; yMax = 4;

 % Grid
 x = linspace(xMin,xMax, N);
 y = linspace(yMin,yMax, N);
 [xx, yy] = meshgrid(x,y);

 f = 1 + 2.*xx.^2 + yy.^2;

 F = simpson2d(f,xMin,xMax,yMin,yMax);
 % Output to Commmand Window
  fprintf('integral  F = %2.10e  \n',F)
```

→ integral F = 5.9200000000e+02 (exact 592)

Example 5 area of a semicircle cemCh3.m (CELL 5)

```
%% CELL 5:   [2D] integration:  simpson2d.m   circle

  clear; close all; clc
% Limits   >>>>>
  N = 999;
  xMin = -1; xMax = 1;
  yMin = -1; yMax = 1;

 % Grid
 x = linspace(xMin,xMax, N);
 y = linspace(yMin,yMax, N);
 [xx, yy] = meshgrid(x,y);

 f = ones(N,N);
 f((xx.^2 + yy.^2) > 1) = 0;

 F = simpson2d(f,xMin,xMax,yMin,yMax);
 % Output to Commmand Window
  fprintf('integral  F = %2.10e  \n',F)
```

integral F = 3.1414278122e+00 → π

[3D] integrals

We can compute the value of triple (volume) integrals of the form

$$F = \int_{xMin}^{xMax} \int_{yMin}^{yMax} \int_{zMin}^{zMax} f(x, y, z)\, dx\, dy\, dz$$

You can use the Matlab function **integral3**.

Example 6 **cemCh3.m (CELL 6)**

$$F = \int_0^2 \int_0^4 \int_0^8 x^2\, y^3\, z^4\, dx\, dy\, dz$$

```
%% CELL 6:   [3D] integration

 clear; close all; clc

 xmin = 0; xmax = 2;
 ymin = 0; ymax = 4;
 zmin = 0; zmax = 8;

 fun = @(x,y,z) x.^2.*y.^3.*z.^4;
 F = integral3(fun,xmin,xmax,ymin,ymax,zmin,zmax);

 fprintf('integral  F = %2.10e  \n',F)
  →  integral  F = 1.1184810667e+06
                      (exact 1.118481066666667e+06)
```

Example 7 **cemCh3.m (CELL 7)**

$$f(x,y,z) = x\cos(y) + x^2\cos(z)$$

Integral Over the Unit Sphere in Cartesian Coordinates

```
%%  [3D] integration

 clear; close all; clc

 xmin = -1;
 xmax = 1;
 ymin = @(x)     -sqrt(1 - x.^2);
 ymax = @(x)      sqrt(1 - x.^2);
 zmin = @(x,y)   -sqrt(1 - x.^2 - y.^2);
 zmax = @(x,y)    sqrt(1 - x.^2 - y.^2);

 fun = @(x,y,z) x.*cos(y) + x.^2.*cos(z);
 F = integral3(fun,xmin,xmax,ymin,ymax,zmin,zmax);

 fprintf('integral  F = %2.10e  \n',F)
```

→ integral F = 7.7955545466e-01

LINE INTEGRALS

In electromagnetism the line integral is often encountered. A **line integral** is an expression of the form

$$F = \int_{a_{path}}^{b_{path}} \vec{v} \bullet d\vec{L}$$

where $\vec{v}$ is a vector function, $d\vec{L}$ is an infinitesimal displacement vector and the integral is performed along a prescribed path from a to b. If the path is a closed loop (a = b) then we write

$$F = \oint \vec{v} \bullet d\vec{L}$$

At each point on the path, we take the dot product of $\vec{v}$ (evaluated at that point) with the displacement $d\vec{L}$ to the next point.

There are some special vectors, that the value of the integral F is **independent** of the integration path.

Length of a curve

Consider a continuous curve $y = f(x, y)$ on the interval [a, b]. Then the length L along of the curve is given by

$$L = \int_a^b ds$$

$$ds^2 = dx^2 + dy^2 \Rightarrow ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \int_{x_1}^{x_2} \sqrt{1 + f\,'(x)^2}\ dx$$

Example 8 Line integral of a scalar function cemCh3 (CELL 8)

The line integral is

$$L = \int_0^{2+4i} \frac{s+1}{s-1-2i}\, ds$$

The function to be integrated has a singularity at the pint s = 1 + 2 I and thus a path must not pass through it.

Path 1: 0 → 2+0i → 2+4i

integral F1 FR = 1.45410e-01 FI =1.85459e+00

integral F2 FR = -4.42859e+00 FI = 8.42859e+00

integral F = F1 + F2 = -4.2831853072e+00 FI = 1.02832e+01

Path 2: 0 → 1.5+I → 2+4i

integral F1 FR = -1.74088e+00 FI =1.46830e+00

integral F2 FR = -2.54230e+00 FI = 8.81489e+00

integral F = F1 + F2 = -4.2831853072e+00 FI = 1.02832e+01

Path 3: 0 → 1+3i → 2+4i

integral F1 FR = 4.74645e+00 FI =-3.96533e+00

integral F2 FR = 3.53673e+00 FI = 1.68214e+00

integral F = F1 + F2 = 8.2831853072e+00 FI = -2.28319e+00

Path 4: 0 → 1+4i → 2+4i

integral F1 FR = 6.13275e+00 FI =-1.57903e+00

integral F2 FR = 2.15044e+00 FI = -7.04152e-01

integral F = F1 + F2 = 8.2831853072e+00 FI = -2.28319e+00

Note: Paths 1 and 2 give the same result as do paths 3 and 4 but the path lengths 3 and 4 are not equal to paths 1 and 2.

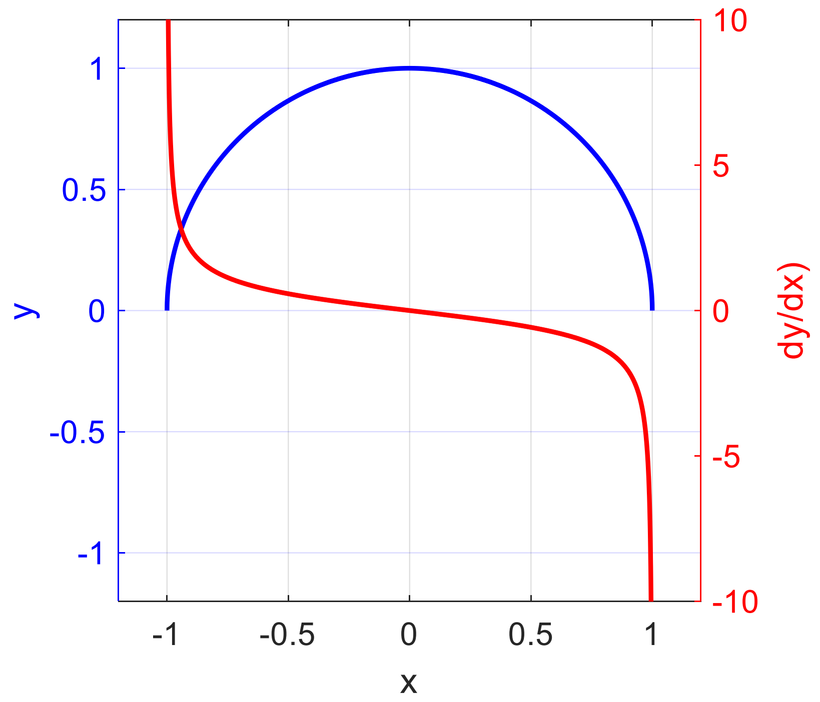
```
%% CELL 8
clear; close all; clc

N = 9999;
% PATH 1
  a = 0; b = 1+4i;
  s = linspace(a,b,N);
  f = ( (s + 1)./(s-1-2*1i) );
  F1 = simpson1d(f,a,b);

  a = b; b = 2 + 4i;
  s = linspace(a,b,N);
  f = ( (s + 1)./(s-1-2*1i) );
  F2 = simpson1d(f,a,b);
  F = F1 + F2;

% Output to Commmand Window
disp('  ')
disp('')
fprintf('integral F1 FR = %2.5e  FI =%2.5e \n',real(F1),
imag(F1))
fprintf('integral F2  FR = %2.5e  FI = %2.5e \n',real(F2),
imag(F2))
fprintf('integral F = F1 + F2 = %2.10e  FI = %2.5e
          \n',real(F),imag(F))
```

Example 9 Length of a semicircle cemCh3.m (CELL 9)



The length of the semicircle of radius 1 is LN = 3.14159 (π)

Line integrals of vector fields

The vector field is given by

$$\vec{V}(x,y,z) = V_x(x,y,z)\,\hat{i} + V_y(x,y,z)\,\hat{j} + V(x,y,z)\,\hat{k}$$

and the [3D] smooth curve is given by

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k} \qquad a \le t \le b$$

The line integral of the vector $\vec{V}$ along the path C is

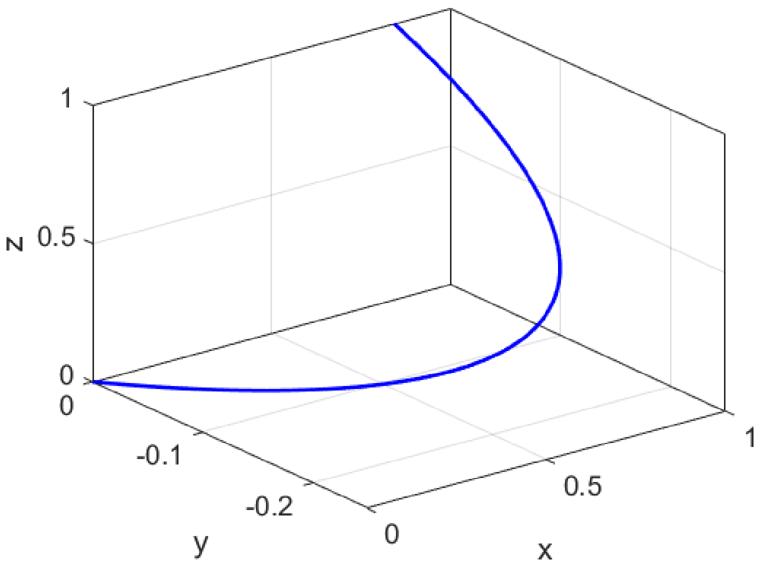
$$\int_C \vec{V} \bullet d\vec{r} = \int_a^b \vec{V}\left(\vec{r}(t)\right) r'(t)\; dt$$

Example 10 cemCh3.m (CELL 10)

The vector is $\vec{V} = \left[x^2 y\,z,\; x\,y,\; 2\,y\,z\right]$

The curve is $\vec{r}(t) = \sin(t)\,\hat{i} + \left(t^2 - t\right)\hat{j} + t\,\hat{k} \quad 0 \le t \le 1$

→ line integral L = -2.020078e-01




```
%% CELL 10
    clear; close all; clc
    tMin = 0; tMax = 1; N = 999;
    t = linspace(tMin,tMax,N);
    dt = t(2) - t(1);
    x = sin(t);   y = t.^2 - t;   z = t;
% Vector function
    V = [x.^2.*y.*z;  x.*y;  2.*y.*z];
    r = [x; y; z];
% Smooth [3D] curve
    rDash = gradient(r,dt);
% Curve gradient
    VdotDash = dot(V,rDash);
% Line Integral
    L = simpson1d(VdotDash, tMin, tMax);
% Output
    fprintf('line integral  L = %2.6e  \n',L)
% GRAPHICS
    figure(1)
    plot3(x,y,z,'b','LineWidth',2)
    xlabel('x'); ylabel('y'); zlabel('z');
    grid on; box on
    set(gca,'fontsize',14)
```

An important line integral is known as the **fundamental theorem of calculus** where the integral of the gradient ∇f along a path $\hat{L}$ from point a to point b does not depend upon the path from a to b.

$$\int_a^b \nabla f \bullet d\vec{L} = f(b) - f(a)$$

The result is just the difference between f values at the ends, regardless of the path of integration.

First suppose that $\vec{F}$ is a continuous vector field in some domain

D, Then $\vec{F}$ is a **conservative vector field** if there is a function f such that

$$\vec{F} = \nabla f$$

The function f is called a **potential function** for the vector field.

DIVERGENCE (GAUSS'S or GREEN'S) THEOREM

Flux describes any effect that appears to pass or travel (whether it actually moves or not) through a surface.

The flux Φ of a vector $\vec{F}$ is

$$\Phi = \oint \left(\vec{F} \bullet \hat{n}\right) dA$$

The symbol $\oint$ implies the integration over a closed surface which encloses a volume. Is a $\hat{n}$ **unit vector** that is perpendicular to an area element dA.

The flux is a scalar. If $\Phi > 0$ then there is a net flow out of the volume across the surface (source) and if $\Phi < 0$ then there is a net flow into the volume across the surface (sink).

The **divergence theorem** (**Gauss's or Green's theorem**) is

$$\Phi = \oint \left(\vec{F} \bullet \hat{n}\right) dA = \iiint \left(\nabla \bullet \vec{F}\right) dv$$

where dv is a volume element.

Example 11 cemCh3.m (CELL 11)

Consider the vector function $\vec{V} = y^2\, i + (2\,x\,y + z^2)\,\hat{j} + 2\,y\,z\,\hat{k}$

Calculate the surface and volume integrals over a unit cube located at the Origin. (Griffiths Example 1.10)

The surface and volume integrals can be computed using Matlab. There is a considerable amount of coding to find the answers to this problem. However, once the Script is written, it is easy to make changes to compute the integrals for other vector functions and bounded regions.

Solution *Inspect the Script to see how the problem was solved*

Results output to Command Window

```
divV = 2*x + 2*y

volume integral (numeric)   FN = 2.020202

volume integral (symbolic)  FS = 2.000000

surface integrals   -0.333   0.333   -0.333   1.333   0.000   1.000

Surface Integral Stot = 2.000000
```

Note the error in the volume integral computed numerically (FN). The number of grid points used was N = 299. Increasing N improves the accuracy but greatly increases the computation time. You need to manually change the function for the variable divV after running the Script once and looking at the function in the Command Window for $\nabla \bullet \vec{V}$.

Example 12 cemCh3.m (CELL 12)

Consider the vector function $\vec{V} = x\,y\,i + 2\,y\,z\,\hat{j} + 3x\,z\,\hat{k}$

Calculate the surface and volume integrals over a cube of length 2 located at the Origin. (Griffiths Problem 1.32)

It is a bit of an effort to write the Script for Example 11, but it only takes a few minutes to change that Script to compute the answers to this problem.

Solution

divV = 3*x + y + 2*z

volume integral (numeric) FN = 48.484845

volume integral (symbolic) FS = 48.000000

surface integrals

0.000 8.000 0.000 16.000 0.000 24.000

Surface Integral Stot = 48.000000

STOKES' THEOREM

$$\oint \vec{F} \bullet d\,\vec{L} = \iint \left(\nabla \times \vec{F} \bullet \hat{n}_A \right) dA \qquad \text{simple loop}$$

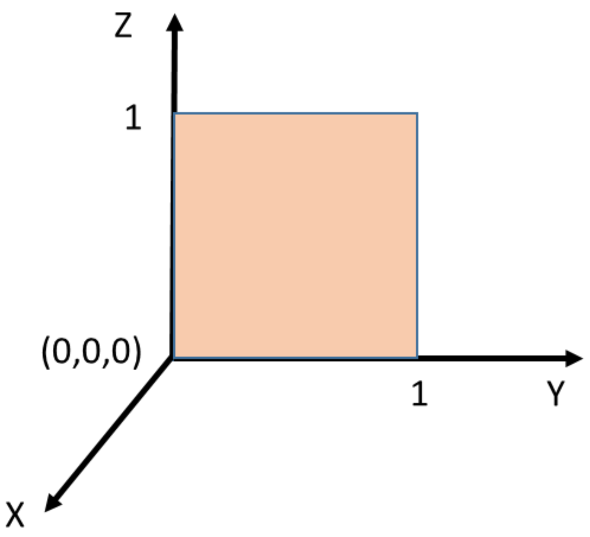
The line integral of the vector $\vec{F}$ about a closed loop is equal to the integral of the curl of $\vec{F}$ over the surface bounded by the loop. $\hat{n}_A$ the normal to an element of the surface.

Example 13 cemStokes.m (CELL 1)

The vector $\vec{V}$ is given by $\vec{V} = 0\hat{i} + \left(2xz + 3y^2\right)\vec{j} + 4yz^2\vec{k}$

Calculate $S_A = \iint \left(\nabla \times \vec{V} \bullet \hat{n}_A\right) dA \qquad S_L = \oint V \bullet d\vec{L}$

for the square surface shown. Check Stokes' theorem.



Griffiths (Example 1.11)

Solution output to Command Window

Curl grad(V x n)

Dx = 4*z^2 - 2*x   Dy = 0   Dz = 2*z

Surface integral SA = 1.333333

Line integral: individual paths

1.000000   1.333333   -1.000000   -0.000000

Line integral SLtot = 1.333333

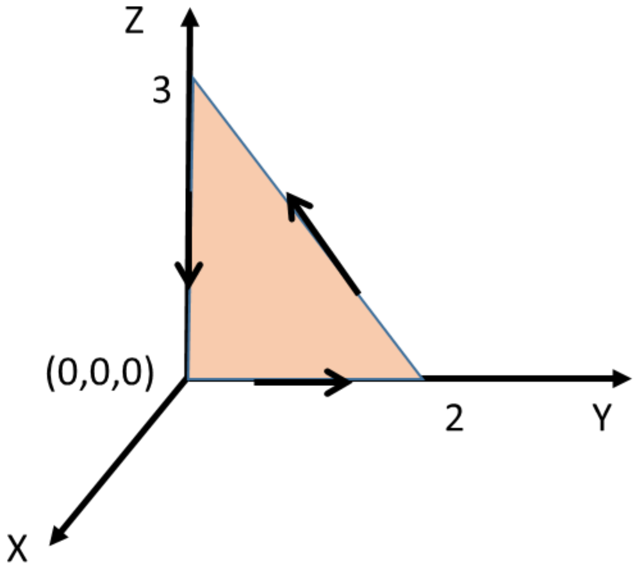
Check the Script to investigate the solution procedure. The parameters for the problem are entered in different sections of the Script. Both symbolic and numeric computations are used.

Example 14 cemStokes.m (CELL 2)

The vector $\vec{V}$ is given by $\vec{V} = x\,y\,\hat{i} + 2\,y\,z\,\vec{j} + 3\,x\,z\,\vec{k}$

Calculate $S_A = \iint \left(\nabla \times \vec{V} \bullet \hat{n}_A \right) dA \qquad S_L = \oint V \bullet d\,\vec{L}$

for the triangular shown. Check Stokes' theorem.



Griffiths (Problem 1.34)

Solution output to Command Window

Line integral: individual paths 0.000000 0.000000 -4.000000

Line integral SLtot = -4.000000

Curl grad(V x n)

Dx = -2*y Dy = -3*z Dz = -x

Surface integral SA = -4.020904

Comment on Script

$\iint\left(\nabla\times\vec{V}\bullet\hat{n}_A\right)dA$ is computed for a rectangular surface as a NxN matrix curlVdotA. An area NxN matrix **AM** is defined such that each element is equal to 1 if the element corresponds to a grid point within the triangular area, otherwise 0.

```
nA = [1 0 0]; % surface unit vector
curlVdotA = curlx.*nA(1) + curly.*nA(2) + curlz.*nA(3);

AM = ones(N,N);
for cy = 1:N
for cz = 1:N
    if zz(cy,cz) > -(3/2)*yy(cy,cz)+3; AM(cy,cz) = 0;
end
end
end

M = AM.*curlVdotA;
```