DOING PHYSICS WITH PYTHON

NONLINEAR [2D] DYNAMICAL SYSTEMS FIXED POINTS, STABILITY ANALYSIS, BIFURCATIONS

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cs120.py to cs133.py

INTRODUCTION

Dynamics of [2D] systems are vast and their behaviours are determined by the nature of equilibrium points, periodic orbits, limit cycles, etc. Critical values for bifurcations parameters are highly associated with system's time evolution and have physical significances. We will consider a number of examples that show how bifurcation parameters are a deciding factor for systems undergoing bifurcation solutions.

An equilibrium or fixed point of a dynamical system generated by system of ordinary differential equations (ODEs) is a solution that does not change with time. The stability of typical equilibria of smooth ODEs is determined by the sign of real part of eigenvalues of the Jacobian matrix. These eigenvalues are often referred to as the eigenvalues of the equilibrium. In [2D] systems the Jacobian matrix is

$$\mathbf{J}(x,y) = \begin{pmatrix} \partial f / \partial x & \partial f / \partial y \\ \partial g / \partial x & \partial g / \partial y \end{pmatrix}$$

It has two eigenvalues, which are either both real or complex-conjugate. The eigenvalues can be found using the Python function **eig**.

Python

The eigenvalues are calculated in Python using the function eig

```
# Jacobian matrix and eigenvalues

J = np.array([[0,0],[0,-1]])

Jev, Jef = eig(J)
```

In [1D] the stability of fixed points is characterised by $f'(x_e)$, where

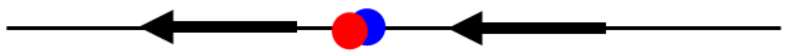
$$f'(x_e) > 0 \quad \Rightarrow \quad x \text{ grows exponentially}$$

$$f'(x_e) < 0 \quad \Rightarrow \quad x \text{ decays}$$

$$f'(x_e) = 0 \quad \Rightarrow \quad \text{indeterminate – check by graphical means}$$

$|f'(x_e)|$ determines the rate of growth or decay and $1/|f'(x_e)|$ gives the characteristic time scale for the growth or decay.

Some fixed points may be semi-stable (half-stable)



The solution to the governing ODEs may not be unique and their maybe many solutions.

A hyperbolic equilibrium can be a

- **Node** when both eigenvalues are real and of the same sign.
  Stable node: both eigenvalues are real and negative.
  Unstable node: both eigenvalues are real and positive.
- **Saddle** is always unstable when the eigenvalues are real and of opposite signs.
- **Focus** (spiral point) when eigenvalues are complex-conjugate;
  The focus is stable when the eigenvalues have negative real part and unstable when they have positive real part.

In [1D] fixed points can be created or destroyed or destabilized as parameters are varied—but in [2D] same is true of closed orbits as well. Thus, we can begin to describe the ways in which oscillations can be turned on or off.

Using eigenvalues and eigenvectors stability criteria

In many physical processes, a system can be modelled using ordinary differential equations and it is important to know the stability of the solutions that describe the behaviour of the system. Often the stability of the solution can be found from the eigenvalues and eigenvectors of the Jacobian matrix.

After finding this stability, you can show whether the system will be stable and damped (if there is a change, the system will adjust itself and return to steady state); unstable and undamped fluctuations, or unstable system in which the amplitude of the fluctuation is always increasing (such system will not be able to return to steady state). For the undamped situation, the constant fluctuation will be hard on the system and can lead to equipment failure or with the ever-increasing amplitude of the fluctuations catastrophic failure will be the result.

Eigenvalues can be used to determine whether a fixed point (also known as an equilibrium point) is stable or unstable. A stable fixed point is such that a system can be initially disturbed around its fixed point yet eventually return to its original location and remain there. A fixed point is unstable if it is not stable.

The eigenvalues of a system linearized around a fixed point can determine the stability behaviour of a system around the fixed point. The particular stability behaviour depends upon the existence of real and imaginary components of the eigenvalues, along with the signs of the real components and the distinctness of their values. That is, the eigenvalues give us the **local stability** around the fixed point.

Real Eigenvalues (no imaginary parts)

- All Zero Eigenvalues: the system will be unstable
- All Positive Eigenvalues: when all eigenvalues are real, positive, and distinct, the system is unstable. On a phase portrait plot, a point in the vector field with multiple vectors circularly surrounding and pointing out of the point is called a source node.
- All Negative Eigenvalues: when all eigenvalues are real, negative, and distinct, the system is stable. Graphically on a gradient field, there will be a node with vectors pointing toward the fixed point. This is called a sink node.
- Positive and Negative Eigenvalues: the fixed point is an unstable saddle point. A saddle point is a point where a series of minimum and maximum points converge at one area in a vector,

without hitting the point. It is called a saddle point because in [3D] surface plot the function looks like a saddle.

- Repeated Eigenvalues: the stability of the critical point depends on whether the eigenvectors associated with the eigenvalues are linearly independent, or orthogonal. This is the case of degeneracy, where more than one eigenvector is associated with an eigenvalue. In general, the determination of the system's behaviour requires further analysis. For the case of a fixed point having only two eigenvalues, however, we can provide the following two possible cases. If the two repeated eigenvalues are positive, then the fixed point is an unstable source. If the two repeated eigenvalues are negative, then the fixed point is a stable sink.

Complex Eigenvalues

When eigenvalues are of the form $a + b$ j, where a and b are real scalars there are three important case to consider: the real part is positive, or negative, or zero. In all cases, when the complex part of an eigenvalue is non-zero, the system will be oscillatory. the stability of oscillating systems (i.e. systems with complex eigenvalues) can be determined entirely by examination of the real part. Although the sign of the complex part of the eigenvalue may cause a phase shift of the oscillation, the stability is unaffected.

- Positive Real Part: the system is unstable and behaves as an unstable oscillator. This can be visualized as a vector tracing a spiral away from the fixed point. The plot of response with time of this situation would look sinusoidal with ever-increasing amplitude. This situation is usually undesirable as any external disturbance will result in the system itself not returning to the steady state.
- Zero Real Part: the system behaves as an undamped oscillator. This can be visualized in two dimensions as a vector tracing a circle around a point. The plot of response with time would look sinusoidal.
- Negative Real Part: the system is stable and behaves as a damped oscillator. This can be visualized as a vector tracing a spiral toward the fixed point. The plot of response with time of this situation would look sinusoidal with ever-decreasing amplitude. This system is stable since steady state will be reached even after a disturbance to the system. The oscillation will bring the system back to the fixed point. It is important to know that having all negative real parts of eigenvalues is a necessary and sufficient condition of a stable system.

If the phase portrait changes its topological structure as a parameter is varied, then a **bifurcation** has occurred. Examples include changes in the number or stability of fixed points, closed orbits, or saddle connections as a parameter is varied. That is, the qualitative structure of the flow can change as parameters are varied. In particular, fixed points can be created or destroyed, or their stability can change. These qualitative changes in the dynamics are called bifurcations, and the parameter values at which they occur are called bifurcation points. Bifurcations are important scientifically—they provide models of transitions and instabilities as some control parameter is varied.

The evolution of a [2D] system can lead to growth, decay, equilibrium, oscillations, periodic motion, aperiodic motion, and chaos. Often the best way to understand the time evolution of a system is not through the mathematics, but through the visualisation of the orbits (trajectories) in phase space where the vector field can give a qualitative view. Local stability equilibrium at a point is characterised by small disturbances being damped out in time, whereas local instability, the disturbance grows with time. Stable fixed points (equilibrium points) are referred to as attractors or sinks. Unstable fixed points are referred to as repellers or sources.

To analyse a dynamical system, it is important to determine the existence of equilibrium points. For example, the acrobats in the photograph are in a stable equilibrium position: if an acrobat tilts laterally, the long rod moves to causes the system to tilt in the opposite direction, returning to the equilibrium position. If the acrobat did not have the rod, that equilibrium position would be unstable: if an acrobat tilted sideways, then the acrobat would make them tilt further, moving the system away from the equilibrium position.

Example 1 Saddle-Node Bifurcation

The saddle-node bifurcation is the basic mechanism by which fixed points are created and destroyed. As a parameter is varied, two fixed points move toward each other, collide, and mutually annihilate.

Consider the [2D] nonlinear dynamical system governed by the equations

$$\dot{x} = r - x^2 \qquad \dot{y} = -y \qquad f(x) = r - x^2 \quad g(y) = -y$$

where r is the bifurcation parameter. The fixed points of the system are dependent upon the bifurcation parameter r.

We need to consider the three cases when $r < 0$, $r = 0$ and $r > 0$ individually to explore the system dynamics for the x subsystem given the fact that in the y subsystem, the y-direction the motion is exponentially damped $(t \to \infty \Rightarrow y \to 0)$. Figure 1.1 shows the phase portrait plots for $r = 9 > 0$, $r = 0$ and $r = -9 < 0$. For $r > 0$, there are two fixed points $(\sqrt{r}, 0)$ which is a **node** (stable) and $(-\sqrt{r}, 0)$ which is a **saddle** (unstable). As r decreases, the saddle and node move closer and coalesce at $r = 0$ to give a semi-stable fixed point. When $r < 0$, the peak of the parabola falls below zero and all fixed points are annihilated.

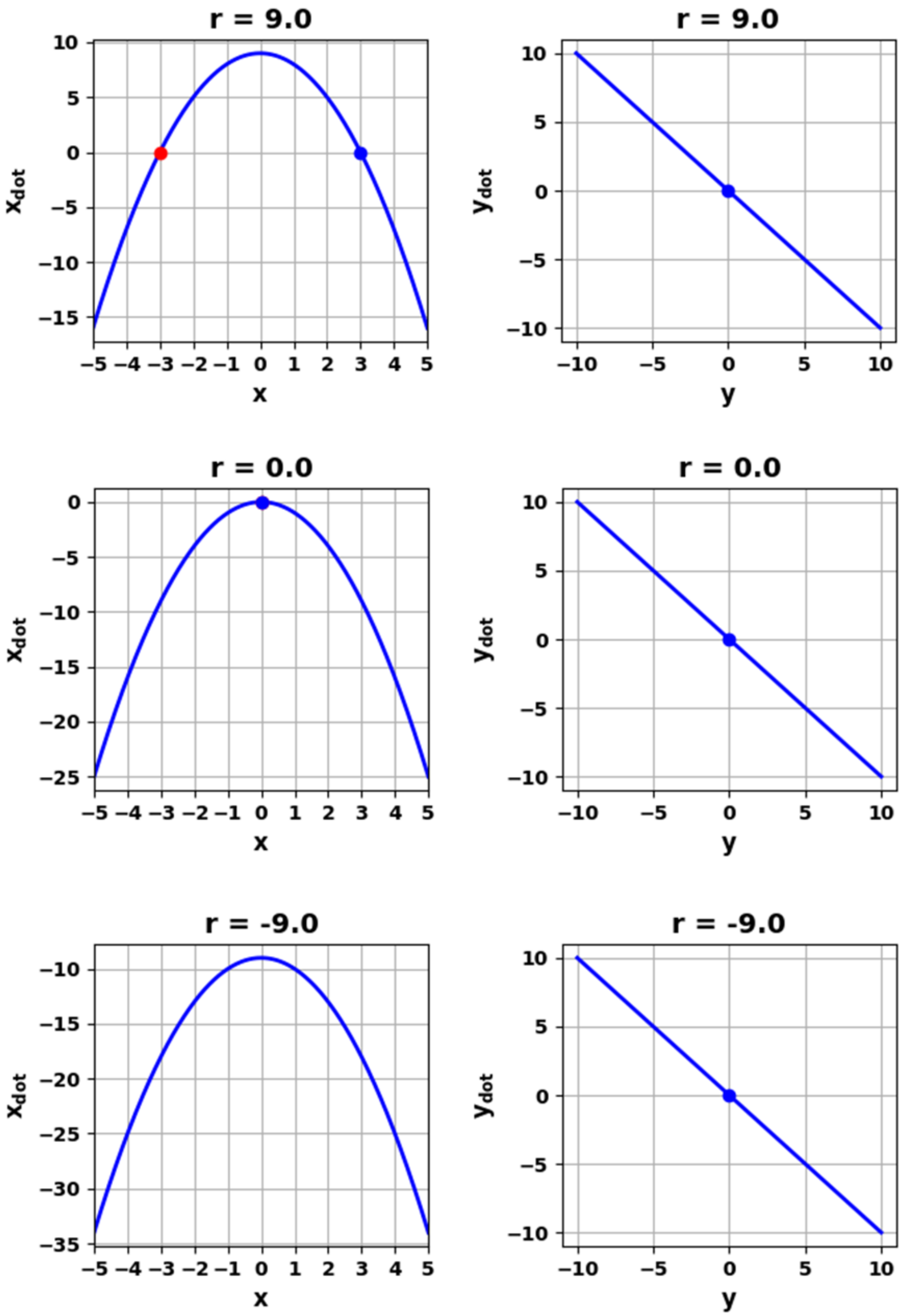


Fig. 1.1.

Mathematical analysis

Fixed points: $\dot{x} = r - x_e^{\;2} = 0 \quad x_e = \pm\sqrt{r}$

Stability: To determine the stabilities of the fixed points, one needs to evaluate the Jacobian matrix of the system for local stability and find the eigenvalues. The Jacobian matrix is

$$\mathbf{J}(x,y) = \begin{pmatrix} \partial f / \partial x & \partial f / \partial y \\ \partial g / \partial x & \partial g / \partial y \end{pmatrix} = \begin{pmatrix} -2x & 0 \\ 0 & -1 \end{pmatrix}$$

***r* > 0** **cs120.py**

The system has two fixed points: $\left(-\sqrt{r},\, 0\right) \quad \left(+\sqrt{r},\, 0\right)$

$$y = 0 \Rightarrow y_e = 0 \quad \dot{x} = 0 \quad \Rightarrow \quad x_e = -\sqrt{r} \quad x_e = +\sqrt{r}$$

The Jacobian matrices are $\mathbf{J_P}$ and $\mathbf{J_M}$

$$x_e = +\sqrt{r} \quad \mathbf{J_P} = \begin{pmatrix} -2\sqrt{r} & 0 \\ 0 & -1 \end{pmatrix} \qquad x_e = -\sqrt{r} \quad \mathbf{J_M} = \begin{pmatrix} +2\sqrt{r} & 0 \\ 0 & -1 \end{pmatrix}$$

Consider the case when $r = 9$, then the fixed points and eigenvalues of the Jacobian are:

- x_e = (-3, 0) eigenvalues = (+6, -1)

  The eigenvalues are real (positive, negative) therefore the fixed point is a **saddle**. and is **unstable**.

- $x_e = (+3, 0)$ eigenvalues = (-6, -1)

  The eigenvalues are real (negative, negative) therefore the fixed point is a **stable node**.

r **= 0** cs121.py

Fixed point $\dot{x} = 0 \quad r = 0 \;\Rightarrow x_e = 0 \qquad \dot{y} = 0 \Rightarrow y_e = 0$

$$\mathbf{J} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

The eigenvalues of the Jacobian are (0, -1) This indicates that the fixed point (0, 0) is **semi-stable** and is a **saddle equilibrium**.

r **< 0** cs122.py

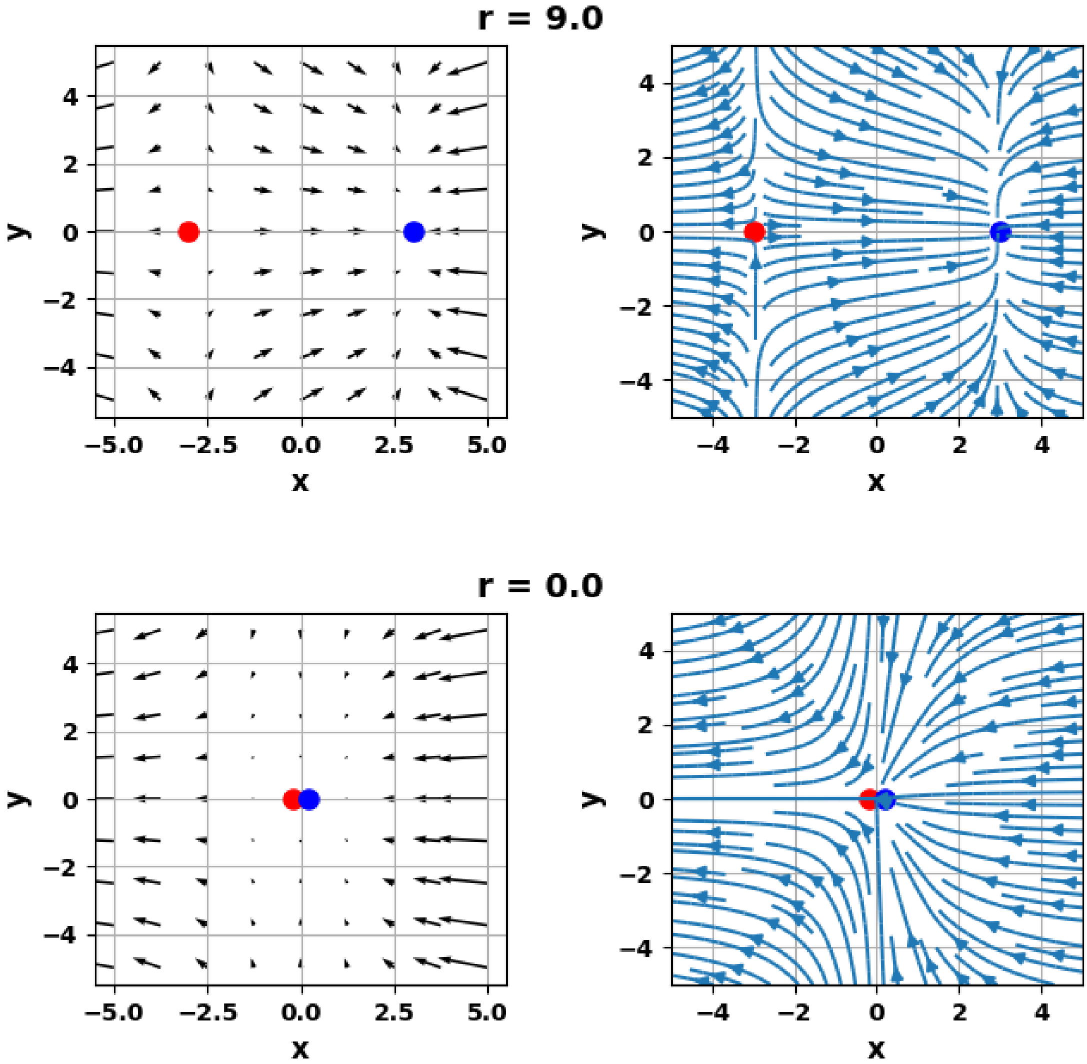
There are no fixed points since $\dot{x} < 0$ for all values of x

$$t \to \infty \quad x \to -\infty$$

Graphical Analysis

Figure 1.2 shows the **vector field** of the system as a Python quiver plot and as a streamplot.

Figure 1.3 shows the time evolution of the system for different initial conditions. The ODEs were solved using the Python function **odeint**.



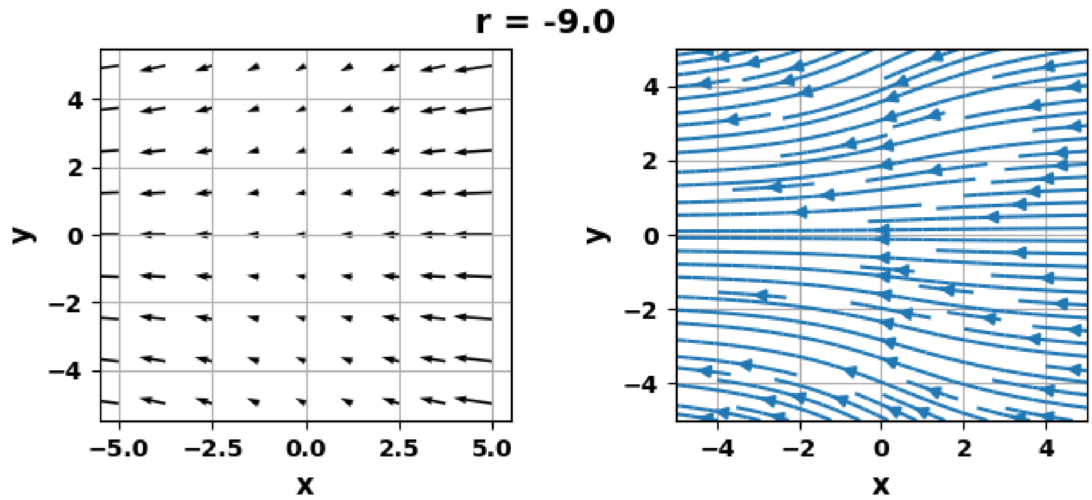
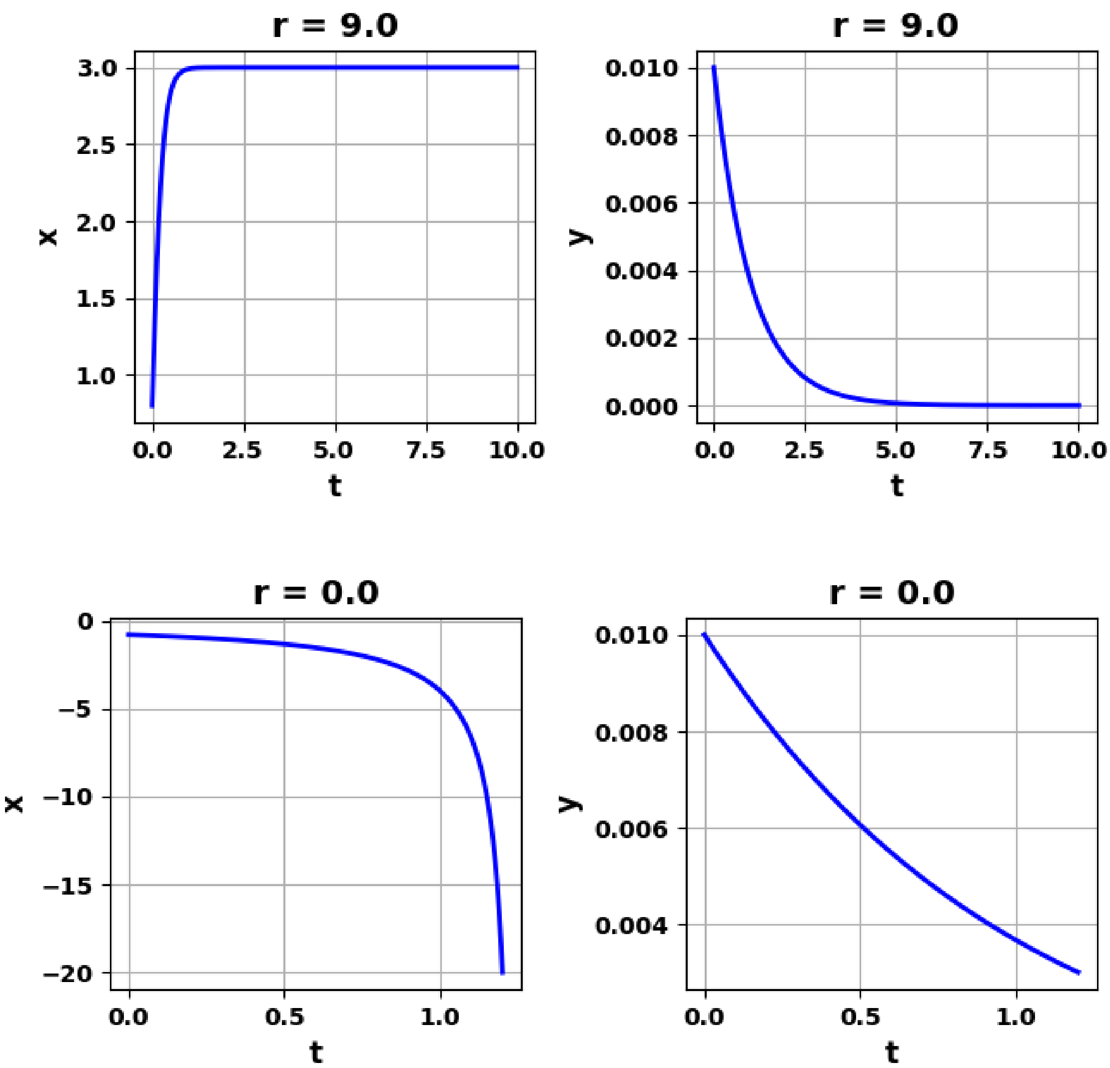


Fig. 1.2.



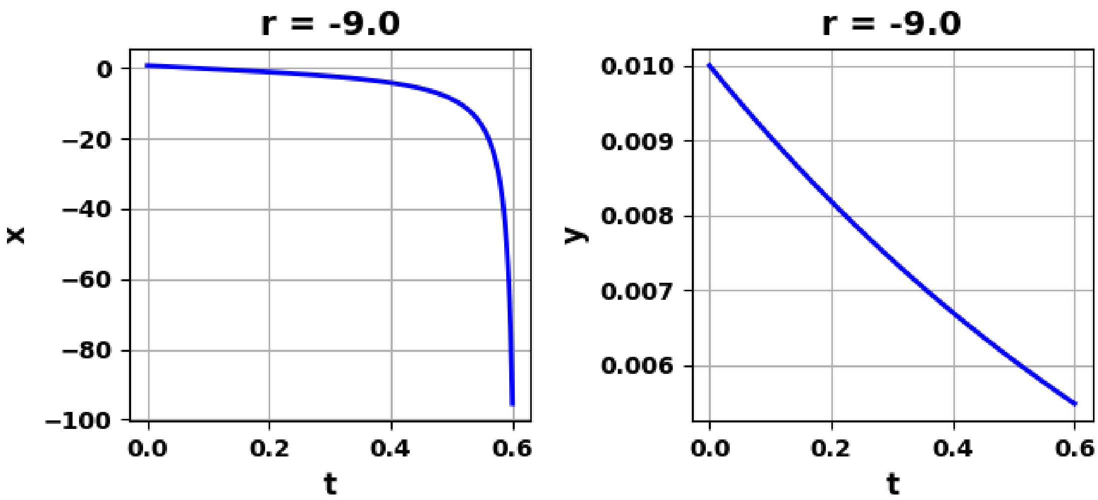


Fig. 1.3

From the graphical analysis, we see that the system with $r > 0$ has two fixed points, one is a stable node $\left(\sqrt{r}, 0\right)$ and the other is a saddle point. When r decreases, the saddle and the stable node approach each other. They collide at $r = 0$ and disappear when $r < 0$. This type of bifurcation is known as **saddle-node bifurcation** (figure 1.4). The name "saddle-node" is because its basic mechanism is the collision of two fixed points - a saddle and a node of the system and in this example the bifurcation point is $r = 0$.

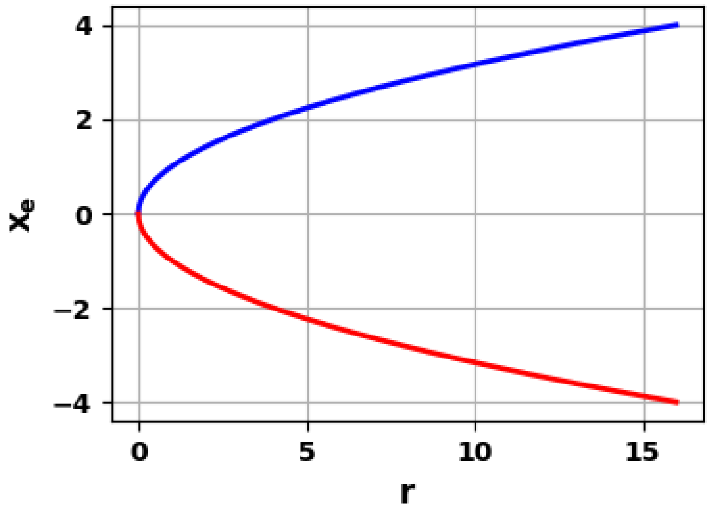


Fig. 1.4.

Even after the fixed points have annihilated each other, they continue to influence the flow as they leave a ghost, a bottleneck region that sucks trajectories in and delays them before allowing passage out the other side. Bifurcation theory is rife with conflicting terminology, and different people use different words for the same thing. For example, the saddle-node bifurcation is sometimes called a fold bifurcation.

Example 2 Transcritical Bifurcation

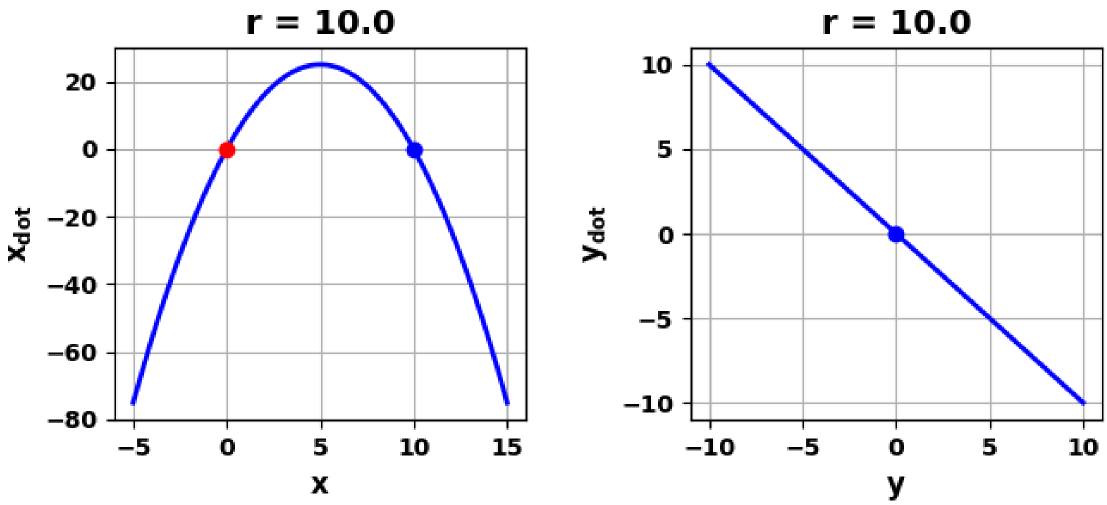
There are certain scientific situations where a fixed point must exist for all values of a parameter and can never be destroyed. For example, in the logistic equation and other simple models for the growth of a single species, there is a fixed point at zero population, regardless of the value of the growth rate. However, such a fixed point may change its stability as the parameter is varied. The **transcritical bifurcation** is the standard mechanism for such changes in stability.

The normal form for a transcritical bifurcation is

$$\dot{x} = r\,x - x^2 \qquad \dot{y} = -y$$

We need to consider the three cases when $r < 0$, $r = 0$ and $r > 0$ individually to explore the system dynamics for the x subsystem given the fact that in the y subsystem, the y-direction the motion is exponentially damped $\left(t \to \infty \Rightarrow y \to 0\right)$.

The fixed points are (-r, 0) and (r, 0) and we see from figure 2.1 that for all values of r, there is a fixed point at the Origin (0, 0).



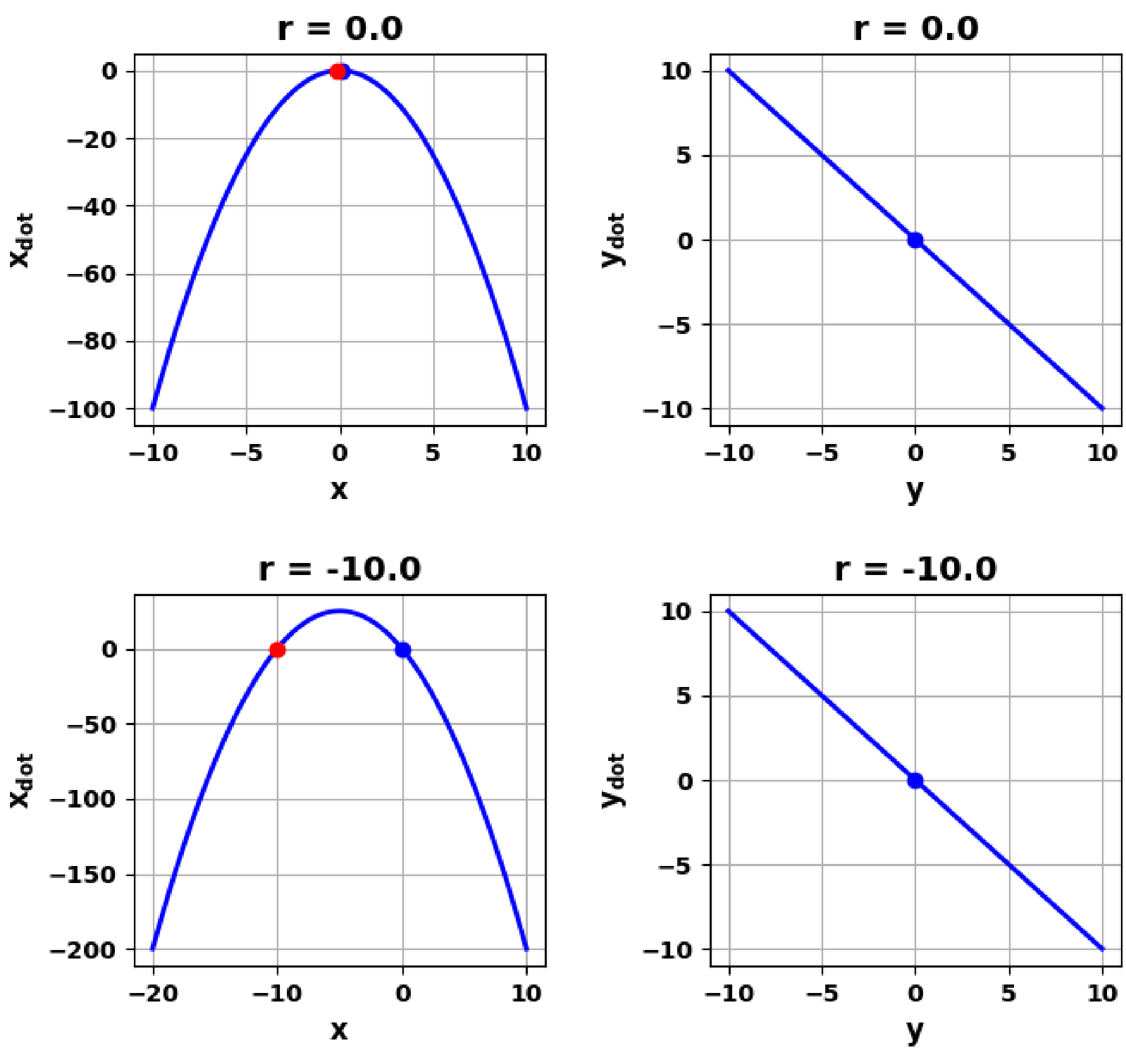


Fig. 2.1

For $r < 0$, there is an unstable fixed point at $x_e = \mathrm{r}$ and a stable fixed point at $x_e = 0$. As r increases, the unstable fixed point approaches the Origin, and coalesces with it when $r = 0$. Finally, when $r > 0$, the Origin has become unstable $x_e = 0$, and $x_e = r$ is now stable. Hence, we can say that an exchange of stabilities has taken place between the two fixed points. Note the important difference between the saddle-node and transcritical bifurcations. In the transcritical case, the two fixed points don't disappear after the bifurcation—instead they just switch their stability.

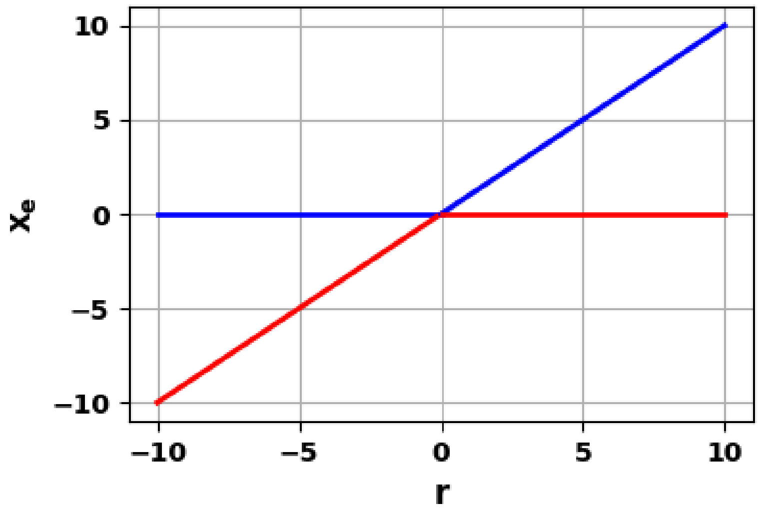


Fig. 2.2

Mathematical analysis

Fixed points: $\dot{x} = r\,x_e - x_e^{\,2} = 0 \quad x_e = 0 \quad x_e = r$

Stability: To determine the stabilities of the fixed points, one needs to evaluate the Jacobian matrix of the system for local stability and find the eigenvalues. The Jacobian matrix is

$$\mathbf{J}(x_e, y_e) = \begin{pmatrix} r - 2x & 0 \\ 0 & -1 \end{pmatrix}$$

***r* > 0** **cs125.py**

The system has two fixed points: $(0,\,0) \quad (r,\,0)$

The Jacobian matrices are

$$x_e = 0 \quad \mathbf{J} = \begin{pmatrix} r & 0 \\ 0 & -1 \end{pmatrix} \qquad x_e = r \quad \mathbf{J} = \begin{pmatrix} -r & 0 \\ 0 & -1 \end{pmatrix}$$

Let r = +10, then the two fixed points are (0, 0) and (10,0) and the Jacobians matrices are

$$\mathbf{J}(0,0) = \begin{pmatrix} 10 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{eigenvalues} = (10, -1)$$

$$\mathbf{J}(10,0) = \begin{pmatrix} -10 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{eigenvalues} = (-10, -1)$$

- The fixed point (0, 0) is an unstable saddle point for $r > 0$.
- The fixed point (r, 0) is stable node for $r > 0$.

r = 0 **cs124.py**

Let r = 0, then there is one fixed points (0, 0)

Jacobian is

$$\mathbf{J}(x, y) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

and the eigenvalues are (0, -1). The fixed point (0, 0) is **semi-stable**.

r < 0 **cs123.py**

Let r = -10, then the two fixed points are (0, 0) and (-10,0) and the Jacobians matrices are

$$\mathbf{J}(0,0) = \begin{pmatrix} -10 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{eigenvalues} = (-10, -1)$$

$$\mathbf{J}(-10,0) = \begin{pmatrix} +10 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{eigenvalues} = (+10, -1)$$

- The fixed point (0, 0) is a stable node for $r < 0$.
- The fixed point (r, 0) is a saddle point (unstable) for $r < 0$.

Graphical analysis

Figure 2.3 shows the **vector field** of the system as a Python quiver plot and as a streamplot. Figure 2.4 shows the time evolution of the system for different initial conditions. The ODEs were solved using the Python function **odeint**.

From the plots in figures 2.3 and 2.4, we see that the behaviour of the system changes when the bifurcation parameter r increases from a negative value and passes through the Origin ($r = 0$), where the saddle becomes a stable node and the stable node becomes a saddle.

$r < 0$ $x_e(0, 0)$ stable node → $r > 0$ $x_e(0, 0)$ unstable saddle

$x_e(r < 0, 0)$ unstable saddle → $x_e(r > 0, 0)$ stable node

This type of bifurcation is known as **transcritical bifurcation**. and the bifurcation point is $r = 0$. This type of bifurcation is same as in a [1D] system where no fixed points are disappeared.

Example 3A Supercritical pitchfork bifurcation

Codes: **cs126.py** ($r < 0$), **cs127.py** ($r = 0$) and **cs128.py** ($r > 0$)

The pitchfork bifurcation is common in physical problems that have a symmetry. For example, many problems have a spatial symmetry between left and right. In such cases, fixed points tend to appear and disappear in symmetrical pairs. For example, consider a vertical beam loaded at the top. For a small load, the beam is stable corresponding to zero horizontal deflection. But, if the load exceeds the buckling threshold, the beam may buckle to either the left or the right. The vertical position has gone unstable, and two new symmetrical fixed points, corresponding to left- and right-buckled configurations, have been born. There are two very different types of pitchfork bifurcation. The simpler type is called supercritical, and will be discussed first.

Consider a [2D] parametric system given by

$$\dot{x} = r\,x - x^3 \qquad \dot{y} = -y$$

Note that this equation for x is invariant under the change of variables $x \rightarrow -x$. That is, if we replace x by $-x$ and then cancel the resulting minus signs on both sides of the equation, the equation does not change. This invariance is the mathematical expression of the left-right symmetry mentioned earlier and the vector fields are equivalent.

Mathematical analysis and graphical analysis

Fixed points: $\dot{x} = 0 \Rightarrow x_e = 0 \quad x_e = \pm\sqrt{r} \quad \dot{y} = 0 \Rightarrow y_e = 0$

The Jacobian of the system is

$$\mathbf{J}(x, y) = \begin{pmatrix} \partial f / \partial x & \partial f / \partial y \\ \partial g / \partial x & \partial g / \partial y \end{pmatrix} = \begin{pmatrix} r - 3x^2 & 0 \\ 0 & -1 \end{pmatrix}$$

***r* < 0** There is only one stable fixed point (0, 0)

Let r = -9 $\quad \mathbf{J}(0,0) = \begin{pmatrix} -9 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{eigenvalues} = (-9, -1)$

***r* = 0** There is only one stable fixed point (0, 0)

Let r = 0 $\quad \mathbf{J}(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{eigenvalues} = (0, -1)$

r > 0 There are three fixed point:

(0, 0) is unstable

$\left(-\sqrt{r}, 0\right)$, and $\left(+\sqrt{r}, 0\right)$ are both stable

Let r = 9 $\quad \mathbf{J}(\pm 3, 0) = \begin{pmatrix} -6 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{eigenvalues} = (-6, -1) \quad$ stable

$$\mathbf{J}(0,0) = \begin{pmatrix} 9 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{eigenvalues} = (9, -1) \qquad \text{unstable}$$

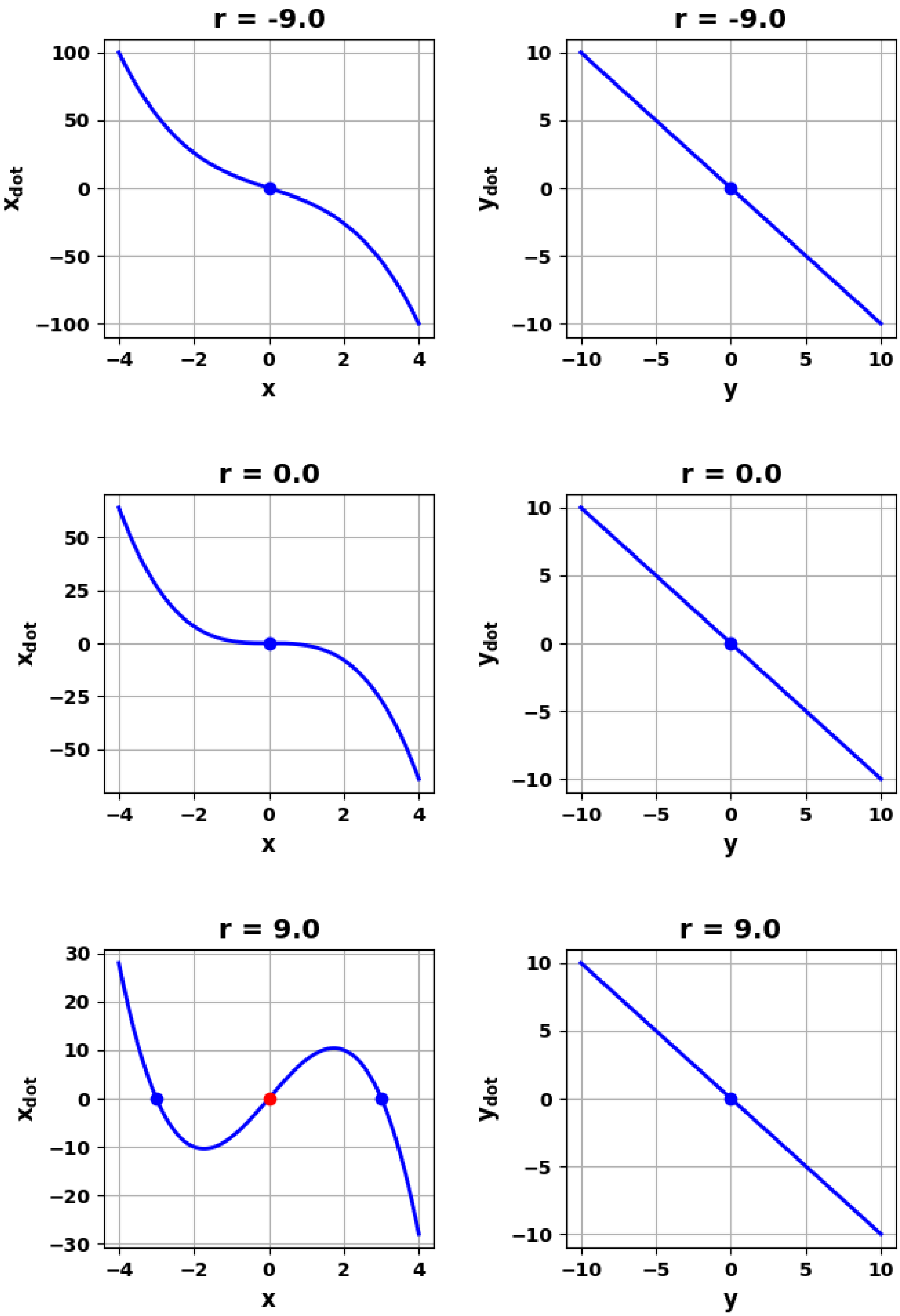


Fig. 3.1 The fixed points of the system for $r < 0$, $r = 0$ and $r > 0$.

Red dot unstable, blue dots are stable

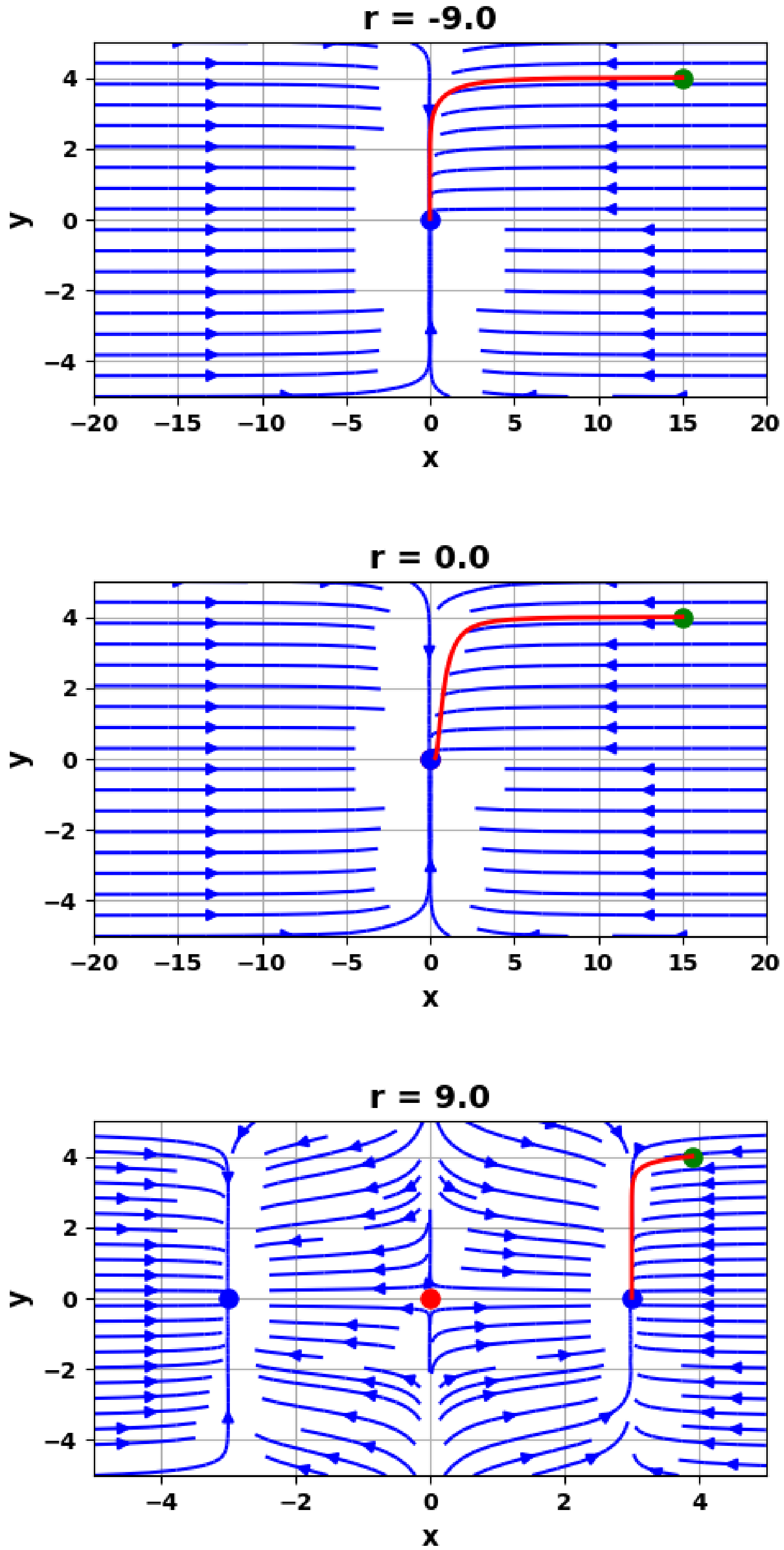


Fig. 3.2 Phase portraits.

When $r < 0$, the Origin is the only fixed point, and it is stable. When $r = 0$, the Origin is still stable, but much weaker. Now solutions no longer decay exponentially fast—instead the decay is a much slower. This lethargic decay is called critical slowing down. Finally, when $r > 0$, the Origin has become unstable. Two new stable fixed points appear on either side of the origin, symmetrically located at $x_e = \pm\sqrt{r}$. The reason for the term "pitchfork" becomes clear when we plot the bifurcation diagram (figure 3.3). Actually, pitchfork trifurcation might be a better word! This type of bifurcation is known as **supercritical pitchfork bifurcation**.

- $r < 0$, the only fixed point is a stable node at the Origin.
- $r = 0$, the Origin is still stable, but now we have very slow (algebraic) decay along the X-direction instead of exponential decay; this is the phenomenon of "critical slowing down.
- $r > 0$, the Origin loses stability and gives birth to two new stable fixed points symmetrically located at $\left(\pm\sqrt{r}, 0\right)$.

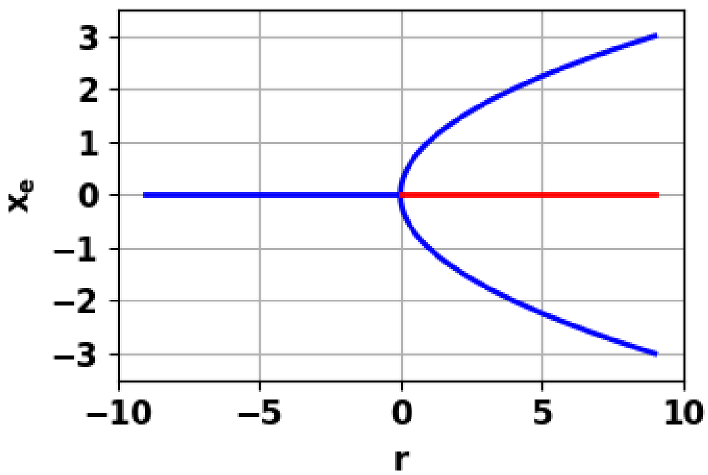


Fig. 3.3

Example 3B Subcritical pitchfork bifurcation

In the supercritical case $\dot{x} = r\,x - x^3 \quad \dot{y} = -y$ (Example 3A), the cubic term is stabilizing and it acts as a restoring force that pulls $x(t)$ back toward $x = 0$. If instead the cubic term were destabilizing, as in

$$\dot{x} = r\,x + x^3 \quad \dot{y} = -y$$

then we'd have a **subcritical pitchfork bifurcation**.

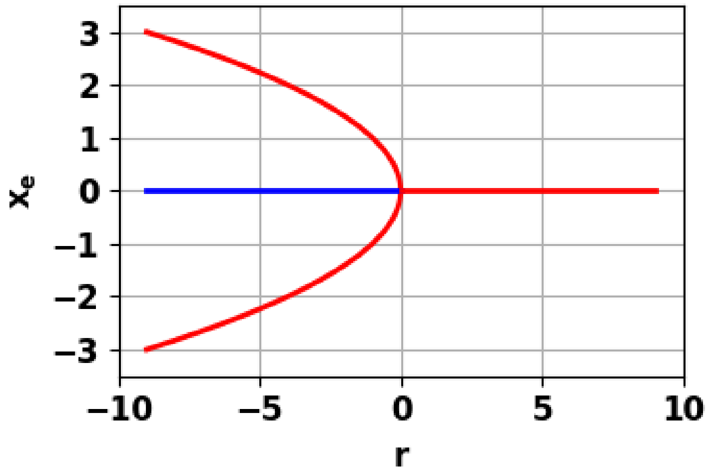


Fig. 3.4

Example 4 Supercritical Hopf Bifurcation cs129.py

Consider a physical system that settles down to equilibrium through exponentially damped oscillations, that is, small disturbances decay after ringing for a period of time. However, suppose that the decay rate depends on a control parameter *r*. In many cases, when the control parameter increases from a negative value, the decay of the oscillations become slower and slower and finally changes to growth at a critical value r_c. For $r > r_c$, the equilibrium state will lose stability and the resulting motion is a small-amplitude, sinusoidal, limit cycle oscillation about the former steady state. This type of bifurcation is known as a **supercritical Hopf bifurcation**.

Consider a two-dimensional system with control (bifurcation) parameter *r*

$$\dot{x} = r\,x - y - x\left(x^2 + y^2\right) \quad \dot{y} = x + r\,y - y\left(x^2 + y^2\right)$$

This is a very simple example of a supercritical Hopf bifurcation.

The system has a unique fixed point at the Origin (0, 0).

The ODEs governing the system are best expressed in polar coordinates where

$$x = R\cos\theta \quad y = R\sin\theta \quad R^2 = x^2 + y^2 \quad \tan\theta = y / x$$

After some tedious algebra, the ODEs in polar coordinates are

$$\dot{R} = r\,R - R^3 \quad \dot{\theta} = 1$$

The ODEs now are decoupled and are easy to analyse for $r < 0$, $r = 0$ and $r > 0$. The fixed points are determined from $\dot{R} = 0$ and the stability from $f(R) = r\,R - R^3 \quad f\,'(R) = r - 3R^2$

$$\dot{R} = 0 \;\Rightarrow\; R = 0 \quad R = \pm\sqrt{r}$$

$$r < 0 \;\Rightarrow\; R = 0 \quad f\,'(0) = r < 0$$

The Origin (0, 0) is the only fixed point which a stable spiral when $r < 0$.

$$r > 0 \;\Rightarrow\; R = 0 \quad f\,'(0) = r > 0$$

The Origin (0, 0) is a fixed point which an unstable spiral when $r > 0$.

$$r > 0 \;\Rightarrow\; R = \sqrt{r} \quad f\,'(\sqrt{r}) = -2r < 0$$

The fixed points with $R = \pm\sqrt{r}$ are stable spirals.

In this example, in terms of the flow in phase space, the supercritical Hopf bifurcation occurs when $r > 0$ ($r > r_c = 0$) when the stable spiral changes into an unstable spiral surrounded by a circular orbit of radius $R = \sqrt{r}$.

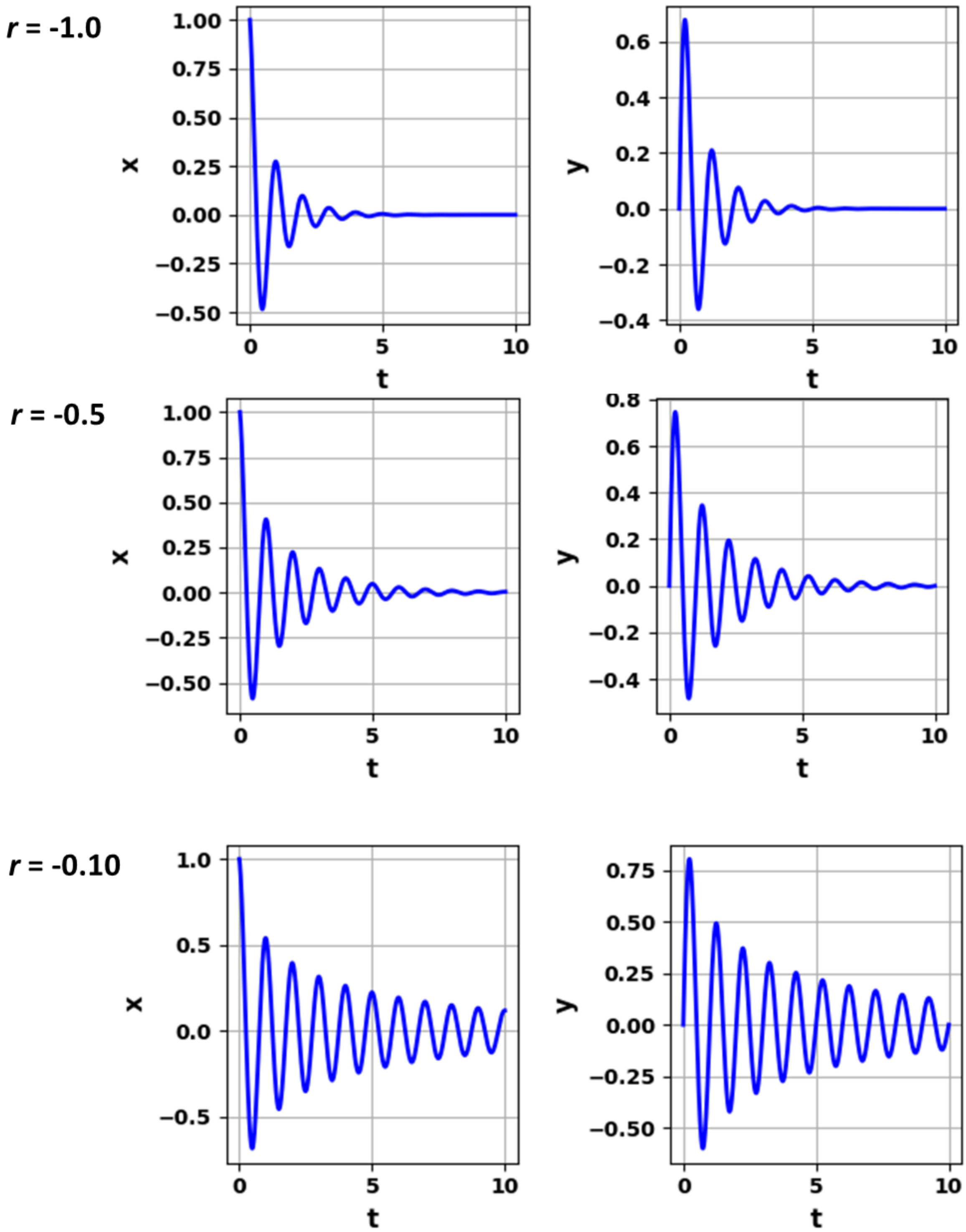


Fig. 4.1 As the control parameter *r* becomes less negative, the rate at which $R \rightarrow 0$ ($x \rightarrow 0$ and $y \rightarrow 0$) decreases.

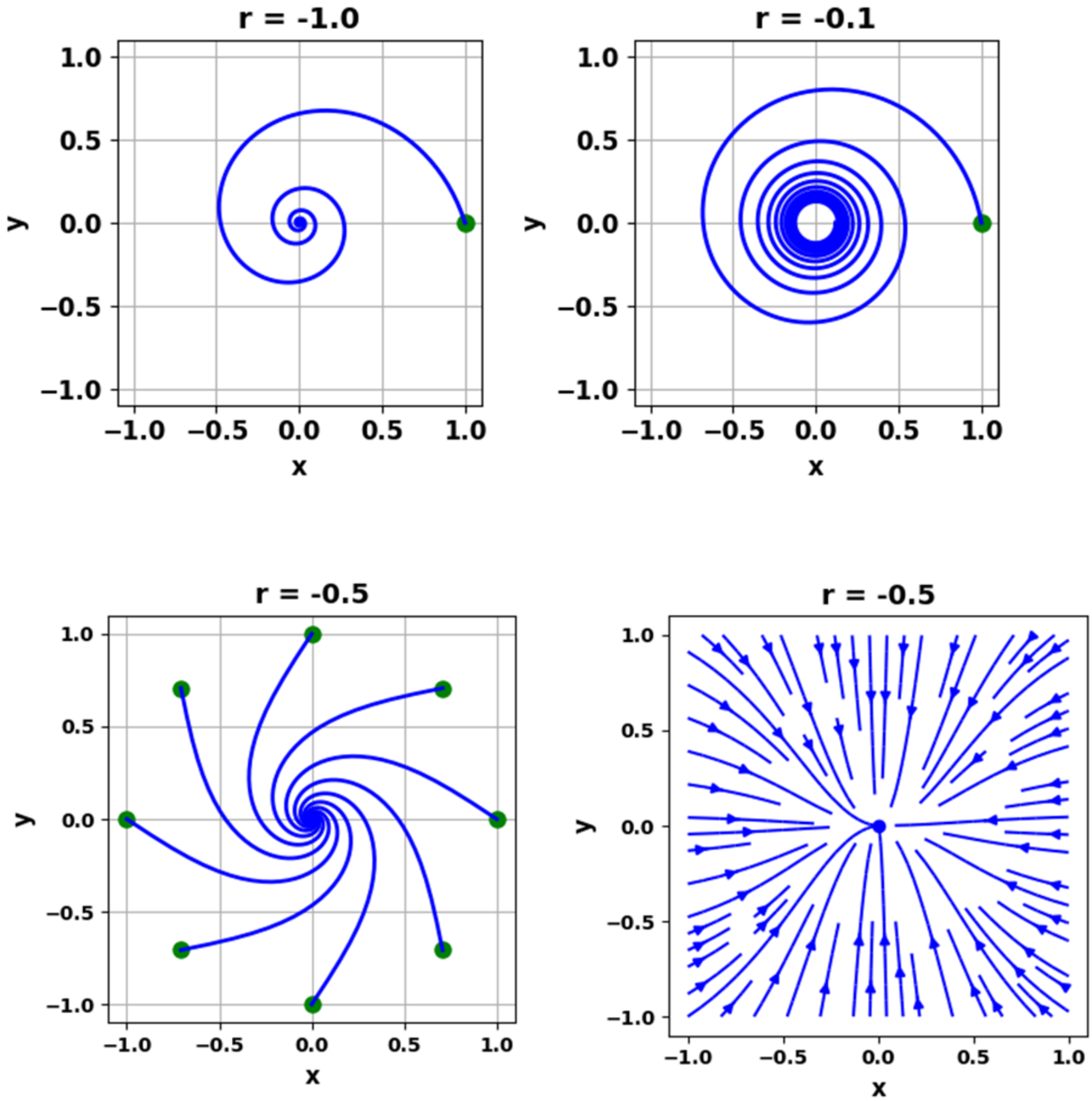


Fig. 4.2 Phase portrait of the system $r < 0$. The fixed point is at the Origin (0, 0) is a **stable spiral** and all trajectories are attracted to it in anti-clockwise direction. The more negative the value of the control parameter r, the more rapidly the oscillation decay to zero.

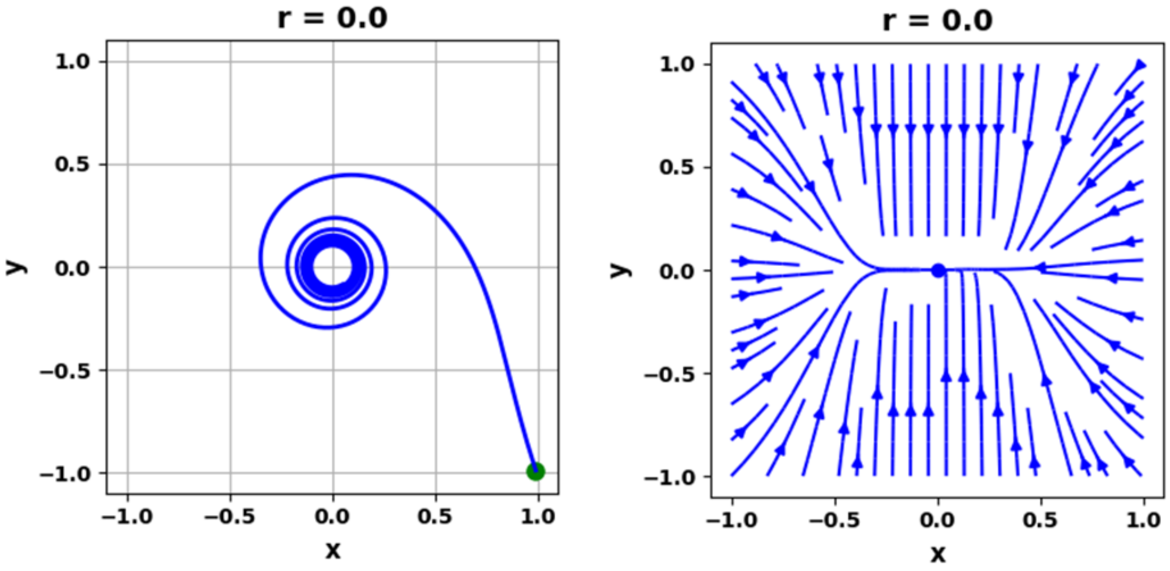
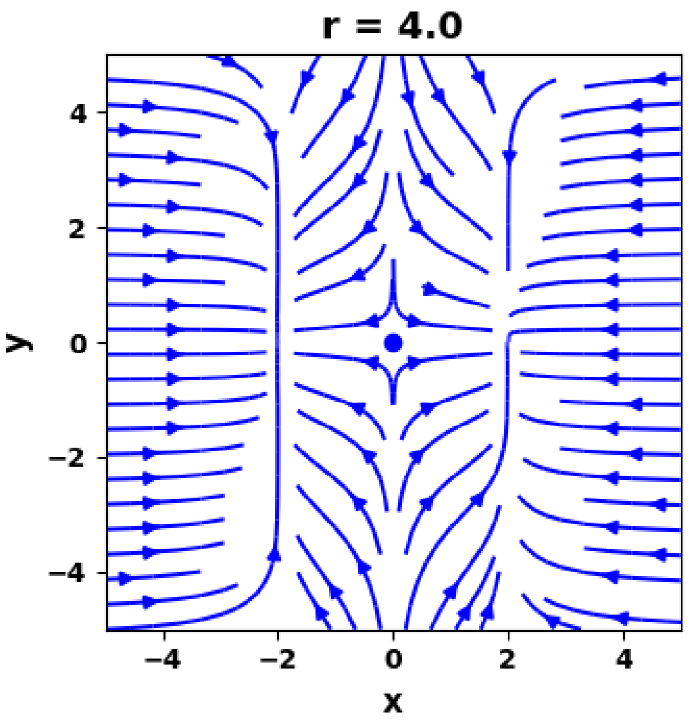


Fig. 4.3 Phase portrait of the system for $r = 0$. The fixed point is at the Origin (0, 0). is a weak **stable spiral** and all trajectories are attracted to it very slowly in anticlockwise direction.

When $r > 0$ the attractor is a stable circular limit cycle with radius $R = \sqrt{r}$.



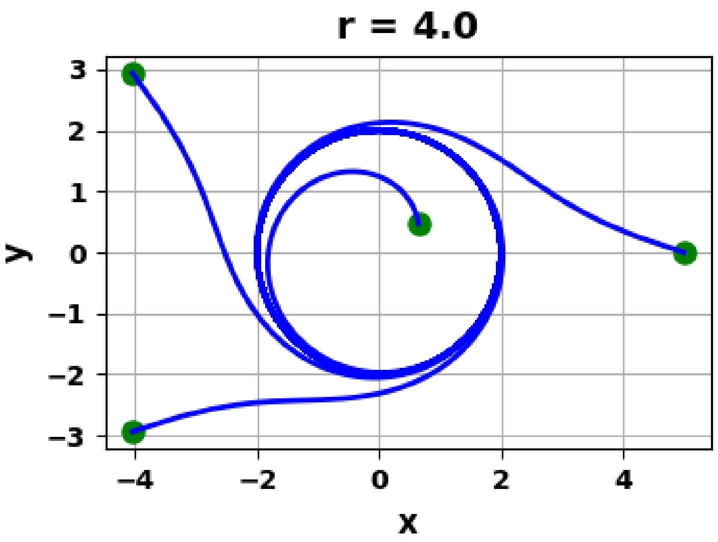


Fig. 4.4 Phase portrait plots showing the limit cycle with radius R

$R = \sqrt{r} \quad R = \sqrt{4} = 2.00$

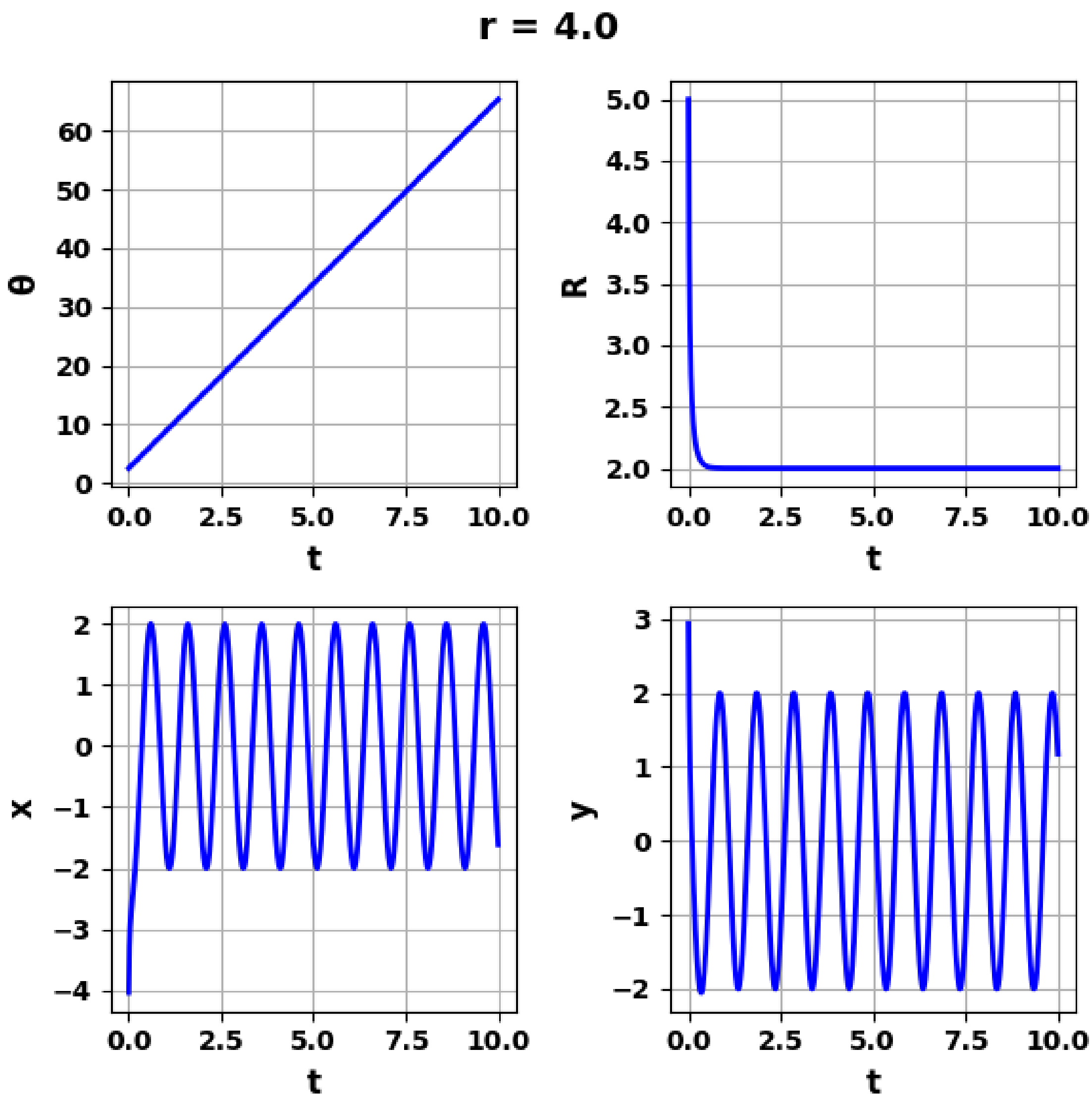


Fig. 4.6 Time evolution of the system $r = 4.0 > 0$

When $r < 0$, the fixed point is at the Origin (0, 0) and is a stable spiral and all trajectories are attracted to it in anticlockwise direction. For $r = 0$, the Origin is still a stable spiral, but is very weak. For r > 0, the Origin is an unstable spiral, and the orbit in phase space is a stable limit cycle of radius $R = \sqrt{r}$.

The Jacobian matrix for the fixed point at the Origin (0, 0) and its eigenvalues are

$$J(0,0) = \begin{pmatrix} r & -1 \\ 1 & r \end{pmatrix} \quad eigenvalues = (r + j,\, r - j) \quad j = \sqrt{-1}$$

The bifurcation point (0, 0) is called a focus or spiral point when eigenvalues are complex-conjugate; The focus is stable when the eigenvalues have negative real part and unstable when they have positive real part. Therefore, the Origin is a stable spiral when $r < 0$ and an unstable spiral when $r > 0$. The eigenvalues cross the imaginary axis from left to right as the parameter r changes from negative to positive values. Hence, a **supercritical Hopf bifurcation** occurs when a stable spiral changes into an unstable spiral surrounded by a limit cycle.

Example 5 Subcritical Hopf Bifurcation cs130.py cs131.py

The Hopf bifurcations come in both super- and subcritical varieties. The subcritical case is always much more dramatic, and potentially dangerous in engineering applications. After the bifurcation, the trajectories must jump to a distant attractor, which may be a fixed point, another limit cycle, or infinity.

Consider a two-dimensional system with bifurcation parameter *r*

$$\dot{x} = r\,x - y - x\left(x^2 + y^2\right) - x\left(x^2 + y^2\right)^2$$

$$\dot{y} = x + r\,y + y\left(x^2 + y^2\right) - y\left(x^2 + y^2\right)^2$$

The system has a unique fixed point at the Origin (0, 0).

The ODEs governing the system are best expressed in polar coordinates where

$$x = R\cos\theta \quad y = R\sin\theta \quad R^2 = x^2 + y^2 \quad \tan\theta = y / x$$

After some tedious algebra, the ODEs in polar coordinates are

$$\dot{R} = r\,R + R^3 - R^5 \quad \dot{\theta} = 1$$

The ODEs now are decoupled and are easy to analyse for $r < 0$, $r = 0$ and $r > 0$.

The important difference from the earlier supercritical case is that the cubic term R^3 is now destabilizing; it helps to drive trajectories away from the Origin.

Mathematical analysis

The fixed points are determined from $\dot{R} = 0$ and the stability from

$$\dot{R} = 0 \;\Rightarrow\; R_e^{\,4} - R_e^{\,2} - r = 0$$

$$R_e^{\,2} = \frac{1 \pm \sqrt{1+4r}}{2}$$

$$f(R) = r\,R + R^3 - R^5 \qquad f\,'(R_e) = r - 3R_e^{\,2} + 5R_e^{\,4}$$

$r < 0$

If **$r < -1/4$** then the Origin (0, 0) is the only fixed point and is a stable spiral.

For **$-1/4 < r < 0$** then there are three fixed point.

$$\dot{R} = 0 \;\Rightarrow\; R_e^{\,4} - R_e^{\,2} - r = 0$$

$$R_e^{\,2} = \frac{1 \pm \sqrt{1+4r}}{2}$$

$$f(R) = r\,R + R^3 - R^5 \qquad f\,'(R_e) = r - 3R_e^{\,2} + 5R_e^{\,4}$$

$$R_e = 0 \qquad f\,'(0) = r = -0.2 < 0 \qquad \text{stable}$$

$$r = -0.2 \quad R_e = 0.526 \quad f\,'(0.526) = 0.247 > 0 \quad \text{unstable}$$

$$R_e = 0.851 \quad f\,'(0.851) = -0.647 < 0 \quad \text{stable}$$

$r > 0$

$$\dot{R} = 0 \Rightarrow R_e^{\,4} - R_e^{\,2} - r = 0$$

$$R_e^{\,2} = \frac{1+\sqrt{1+4r}}{2}$$

$$f(R) = r\,R + R^3 - R^5 \qquad f\,'(R_e) = r - 3R_e^{\,2} + 5R_e^{\,4}$$

$$R_e = 0 \qquad f\,'(0) = r > 0 \qquad \text{unstable}$$

$$r = 0.2 \quad R_e = 1.082 \quad f\,'(1.082) = -3.142 < 0 \quad \text{stable}$$

There are two fixed points when $r > 0$, (0, 0) which is unstable and (R_e, 0) which a stable spiral.

Graphical analysis

The phase portraits shown figure below show:

- $r < 0$ there are two attractors, a stable limit cycle and a stable fixed point at the Origin. Between them lies an unstable cycle. It is the player to watch since as r increases, the unstable cycle tightens like a noose around the fixed point (0, 0). A subcritical Hopf bifurcation occurs at $r = 0$, where the unstable cycle shrinks to zero amplitude and engulfs the Origin, rendering it unstable. **cs130.py**
- $r > 0$ the large-amplitude limit cycle is suddenly the only attractor. Solutions that used to remain near the origin are now forced to grow into large-amplitude oscillations. **cs131.py**

Stable limit cycle

R_{es} =

0.8506508083520399

R = 0.525**8**311121191336

> R_{eu}

Unstable limit cycle

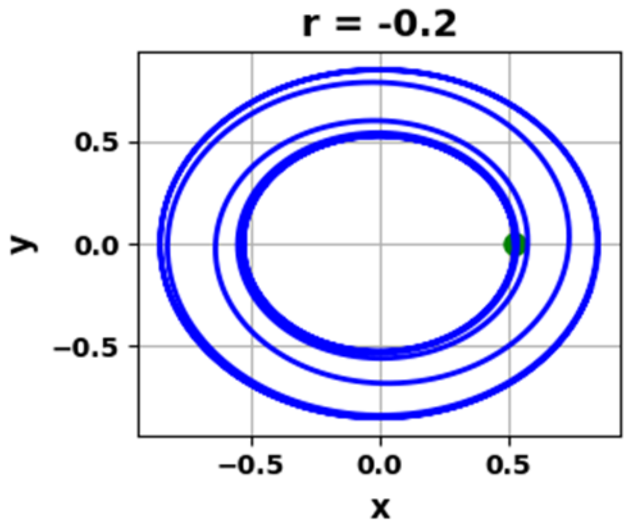
R_{eu} =

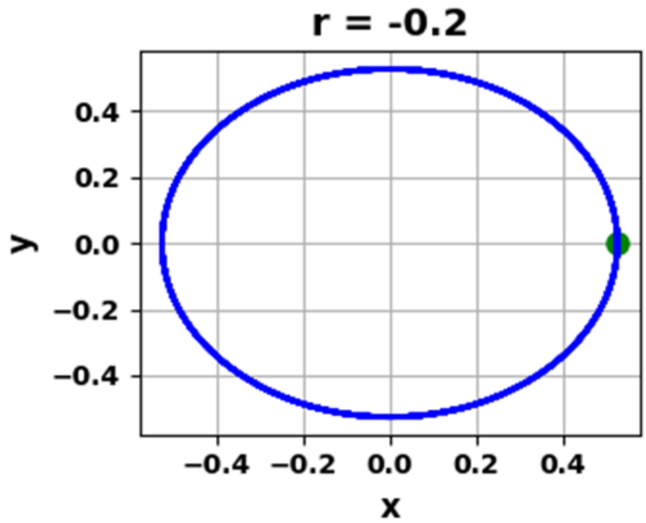
0.525**7**311121191336

R = 0.525**6**311121191336

< R_{eu}

-1/4 < r = -0.20 < 0





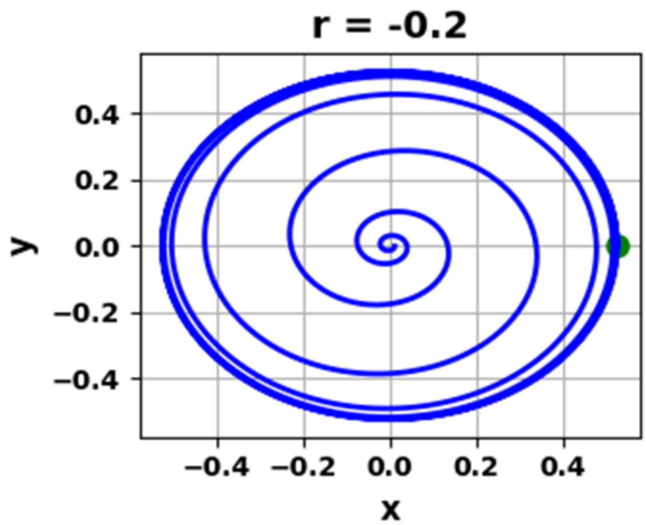


Fig. 5.1 Phase portraits for r = -0.20 for different initial values of R

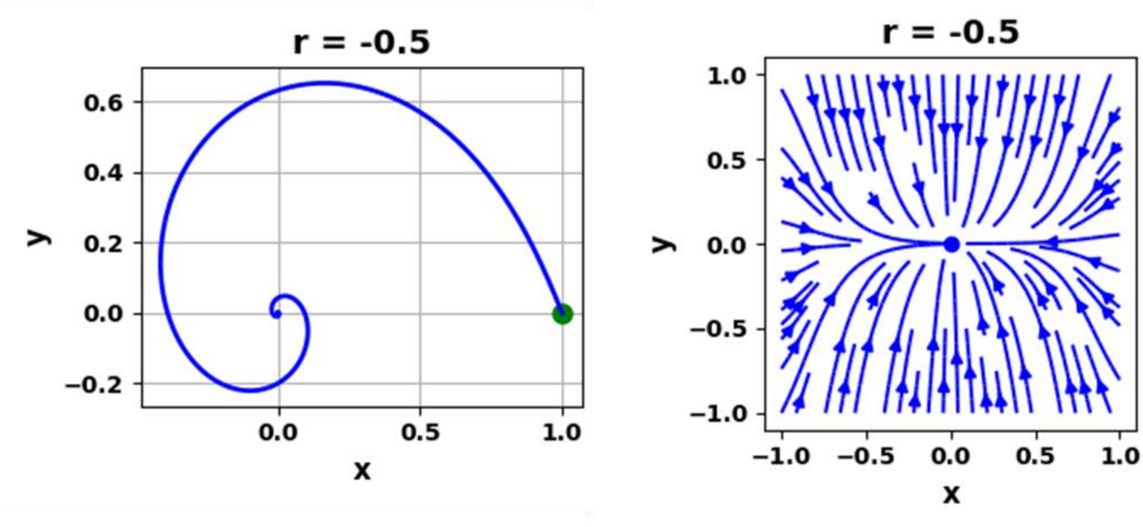


Fig. 5.2 Fig. 5.1. If ***r* < -1/4**, then the only fixed point is the Origin (0, 0) and it is a stable spiral.

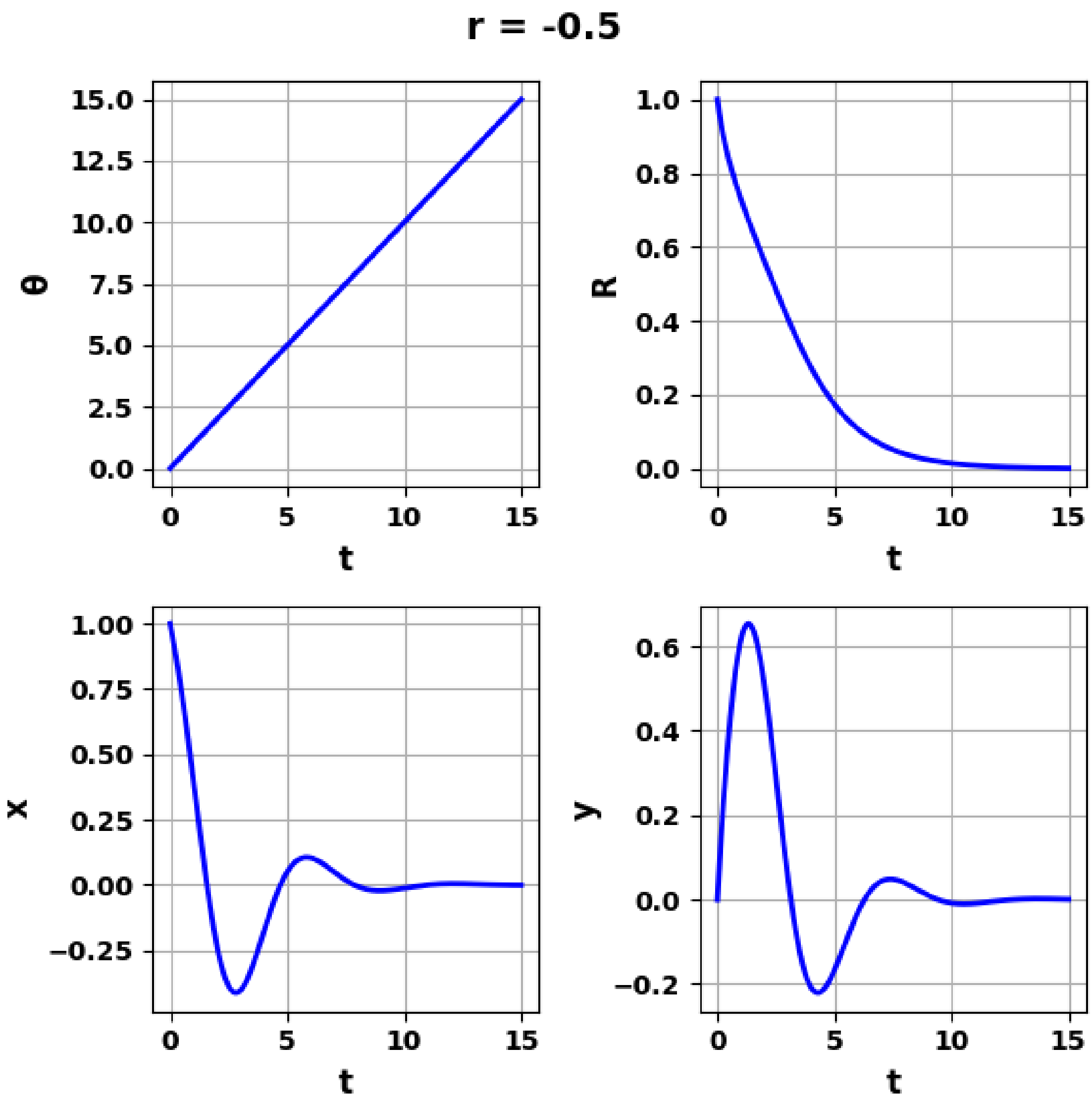


Fig. 5.3 If ***r* < -1/4**, then the only fixed point is the Origin (0, 0) and it is a stable spiral.

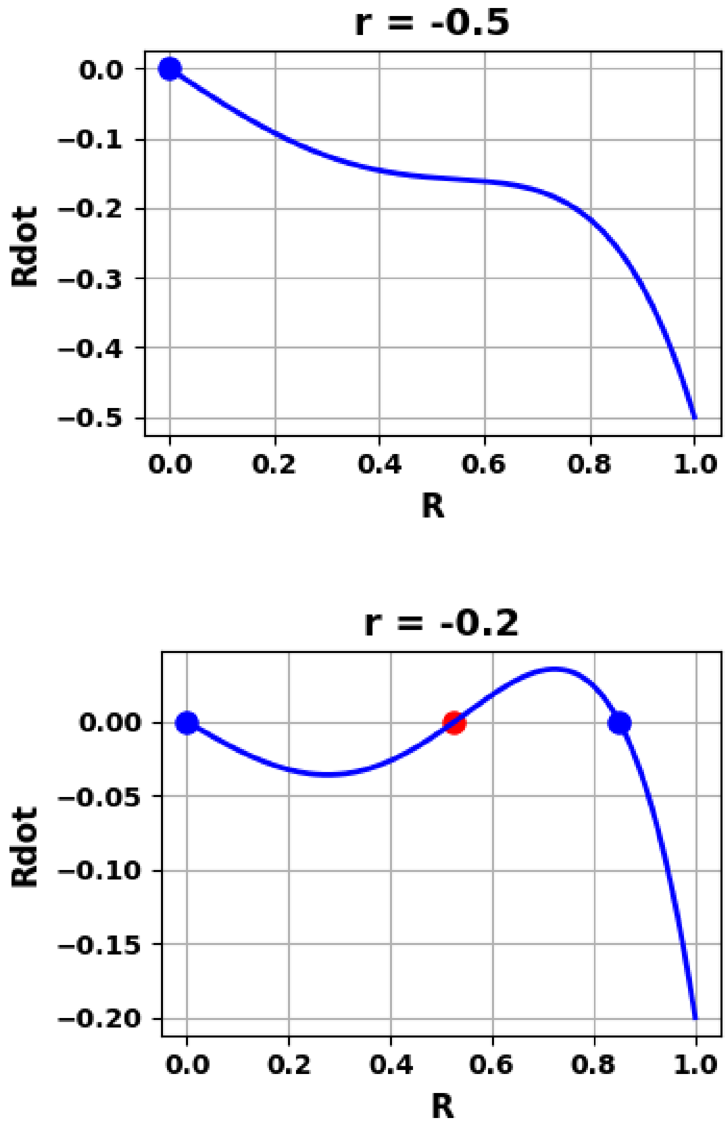


Fig. 5.4 If ***r* < -1/4**, then the only fixed point is the Origin (0, 0) and it is a stable spiral. For **-1/4 < *r* < 0** then there are three fixed point.

blue dots – stable spiral and red dot – unstable spiral.

r > 0 cs131.py

There are two fixed points when $r > 0$, (0, 0) which is unstable and (R_e, 0) which a stable spiral.

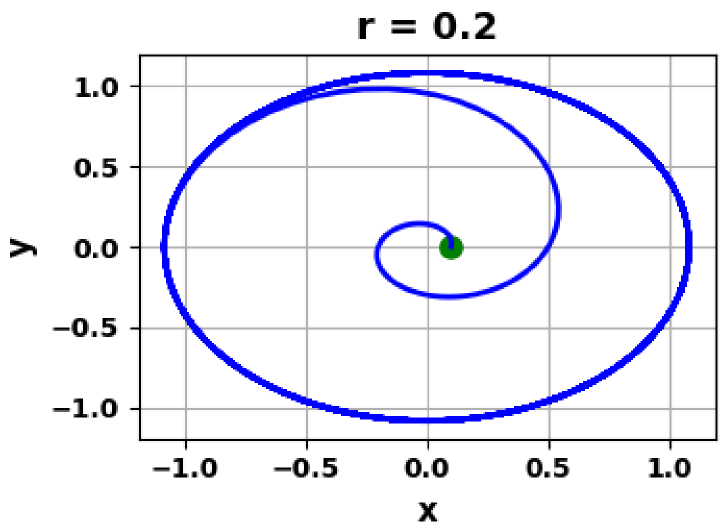


Fig. 5.5 There are two fixed points when $r > 0$, (0, 0) which is unstable and (R_e, 0) which a stable spiral.

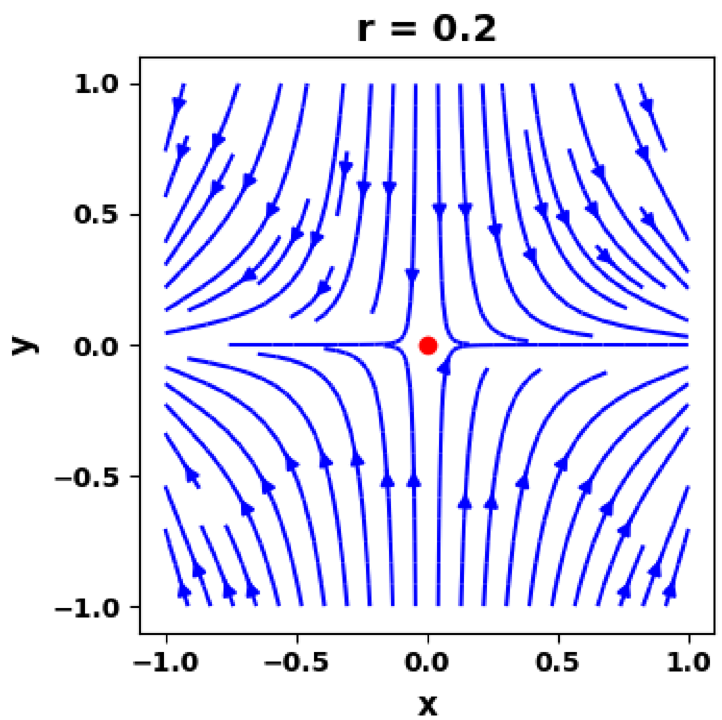


Fig. 5.6 Phase portrait as a streamplot for r = 0.2.

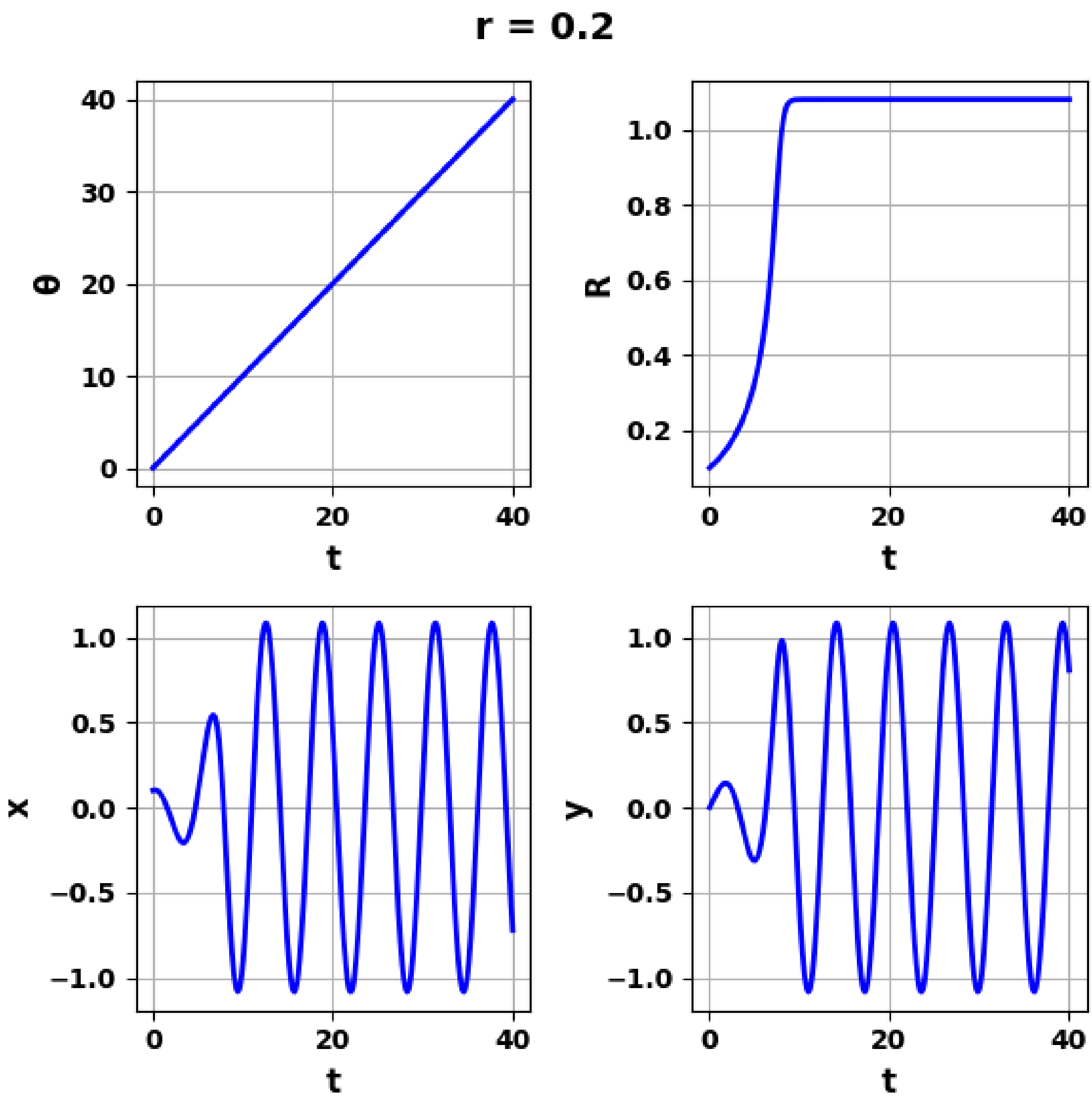


Fig. 5.7 r > 0: The system oscillates with large amplitude.

Note that the system exhibits hysteresis: once large-amplitude oscillations have begun, they cannot be turned off by bringing *r* back to zero. The large oscillations will persist until r = -1/4 where the stable and unstable cycles collide and annihilate.

Example 6 Homoclinic Bifurcations

A homoclinic bifurcation, in this scenario, part of a limit cycle moves closer and closer to a saddle point and at the bifurcation, the limit cycle touches the saddle point and becomes a homoclinic orbit.

Consider the example

$$\dot{x} = y \quad \dot{y} = r\,y + x + x\,y - x^2$$

The fixed points of the system are (-1, 0), (0, 0), and (+1, 0). The fixed point (0, 0) is a saddle and the other two are centres. The phase space orbit is dependent upon the initial conditions (x(0), y(0)).

To study the behaviour of the system, it is best to consider an initial value of r = -0.900 and then run the simulation **cs133.py** for small positive increments in r. The phase space orbit depends upon the initial conditions x(0) and y(0), so they have to be chosen with care. The critical value for the bifurcation parameter is rc ~ -0.865. So, we need to consider values of $r < r_c$ and $r > r_c$.

$r < r_c$

The fixed points and eigenvalues for r = -0.900 are:

(-1,0) ev = [1.02547463 -2.92547463] → unstable

(0, 0) ev = [0.64658561 -1.54658561] → unstable

(+1, 0) ev = [0.05+0.99874922j 0.05-0.99874922j]

→ oscillations: a stable limit cycle and as r increases,

the limit cycle passes closer to a saddle point at the Origin

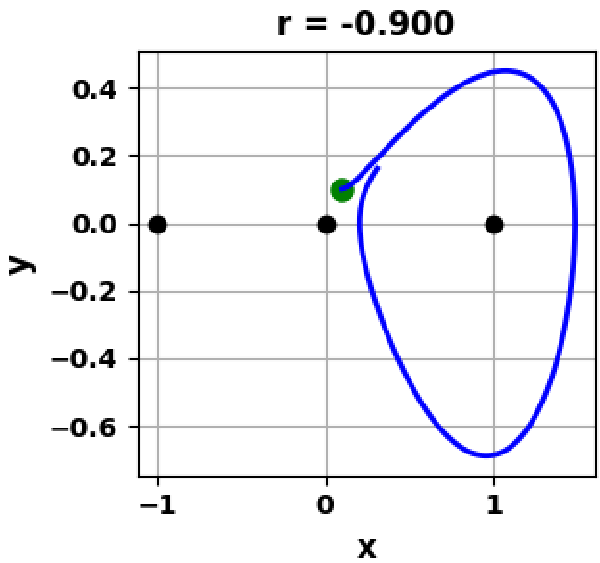
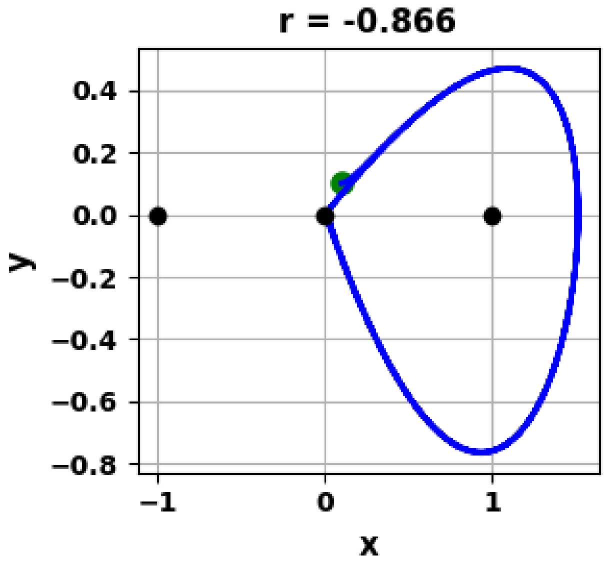


Fig. 6.1 $r < r_c = 0.865$

r ~ r_c



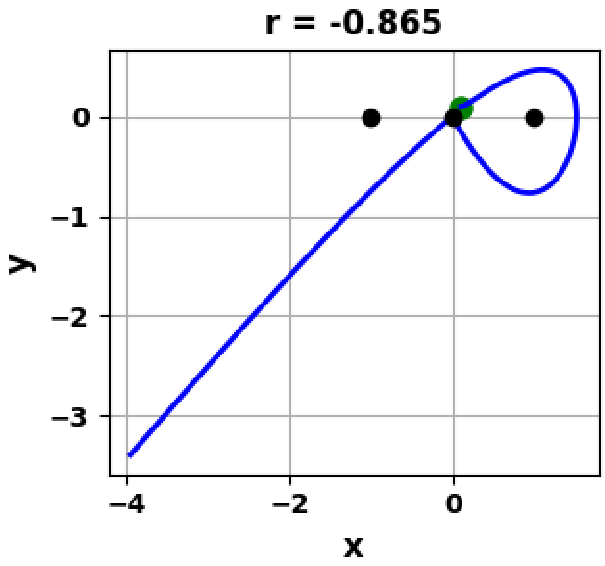


Fig. 6.2 The bifurcation point at (0, 0) occurs when $r_c \sim 0.865$.

$r > r_c$

The fixed points and eigenvalues for r = -0.800 > rc are:

(-1,0) ev = [1.05192213, -2.85192213] → unstable

(0, 0) ev = [0.67703296 -1.47703296] → unstable saddle

(+1, 0) ev = [0.1+0.99498744j 0.1-0.99498744j]

→ the limit cycle swells and breaks the connection to the fixed point at the Origin and the loop is destroyed into the saddle, creating a homoclinic orbit.

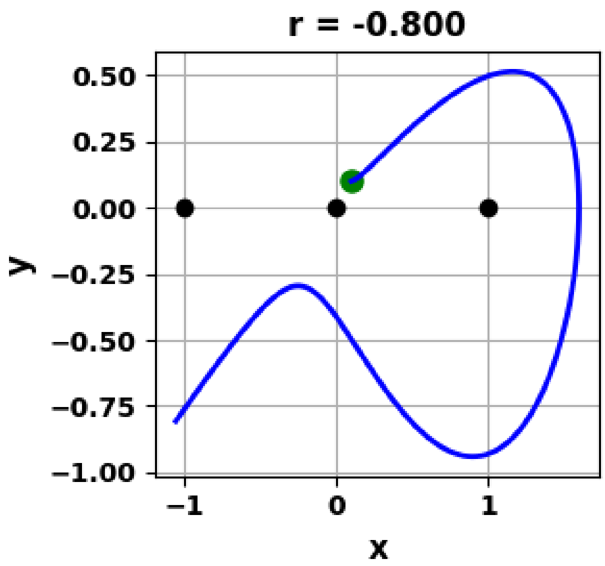


Fig. 6.3 All orbits will go to infinity when $r > r_c$

$(t \to \infty \;\; x \to \pm\infty \;\; y \to \pm\infty)$.

The key to this bifurcation is the behaviour of the unstable manifold of the saddle. Look at the branch of the unstable manifold that leaves the Origin: after it loops around, it either hits the origin veers off to one side or the other.