DOING PHYSICS WITH PYTHON

COMPLEX SYSTEMS

DYNAMICAL SYSTEMS WITH TWO DEGREES OF FREEDOM: PREDATOR-PREY SYSTEMS (Lotka-Volterra Equations)

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DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive

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cs_003.py

Solve predator-prey difference equations / computes the location of the fixed point / finds the peaks in the time evolution plots using the command find_peaks

Plots: R vs t, W vs t, phase space R vs W, quiver R vs W, streamline R vs W, R & W vs t (twin Y-axes)

INTRODUCTION

Consider one species, called the **prey**, which has an ample food supply and the second species, called the **predators**, feeds on the prey. We will consider the example of rabbits as the prey and wolfs as the predator. There are many other species that could have been considered. Our model will have two degrees of freedom with variable R being the number of rabbits at any time and the second variable W for the number of wolfs.

Assume that the principal cause of death among the prey (rabbits R) is being eaten by a predator (wolf W). We also assume that the two species encounter each other at a rate that is proportional to both populations. The model is described by the pair of differential equations

$$
\begin{aligned}
\frac{dR}{dt} &= a_0 R - a_1 R W \\
\frac{dW}{dt} &= -a_2 W + a_3 R W
\end{aligned}
\qquad (1)
$$

where a_0, a_1, a_2, and a_3, are positive constants. The term $-a_1 R W$ decreases the natural growth rate of the rabbits (prey) and the term $+a_3 R W$ increases the natural growth rate of the wolfs (predators).

If there were zero wolfs, then the rabbit population would increase exponentially to infinity and if there were zero rabbits the wolf population would become extinct.

The equations in (1) are known as the **Lotka-Volterra** equations or the **predator-prey** equations.

The Lotka-Volterra equations were proposed as a model to explain the variations in the shark and food-fish populations in the Adriatic Sea by the Italian mathematician Vito Volterra (1860–1940).

To solve the predator-prey equation given by equation 1, we can express the coupled equations as difference equations

$$(2) \qquad \begin{aligned} R(t+\Delta r) &= R(t) + \Delta t\left(a_0 R(t) - a_1\, R(t) W(t)\right) \\ W(t+\Delta r) &= W(t) + \Delta t\left(-a_2 W(t) + a_3\, R(t+\Delta t) W(t)\right) \end{aligned}$$

This pair of coupled difference equation is solved using Python with the code **cs_003.py**.

For our 2 dof dynamical system, it is important to find the fixed-point of the system if they exist. The fixed points give the steady-state values or equilibrium values R_{ss} and W_{ss} (equation 3)

$$R(t+\Delta r) = R(t) \Rightarrow a_0 R_{ss} - a_1 R_{ss} W_{ss} = 0$$

$$(3) \qquad W(t+\Delta r) = W(t) \Rightarrow -a_2 R_{ss} - a_{23} R_{ss} W_{ss} = 0$$

$$\Rightarrow R_{ss} = W_{ss} = 0 \;\; or \;\; R_{ss} = a_2 / a_3 \quad W_{ss} = a_0 / a_1$$

The trivial fix-point is where the two populations become extinct $\left(R_{ss} = 0, W_{ss} = 0\right)$. The non-trivial fixed-point is $\left(R_{ss} = a_2 / a_3,\; W_{ss} = a_0 / a_1\right)$.

SIMULATIONS

Equation 2 is solved using the code **cs_003.py** using the default parameters:

Initial rabbit population $R(0) = 1000$

Initial wolf population $W(0) = 40$

Constants $a_0 = 8.0\text{x}10^{-2}$ $a_1 = 1.0\text{x}10^{-3}$

$a_2 = 2.0\text{x}10^{-2}$ $a_3 = 2.0\text{x}10^{-2}$

Time span (months) $t_{Min} = 0$ $t_{Max} = 500$ $N = 9999$

From equation 3, the equilibrium populations (default parameters) are

$R_{ss} = 1000$ and $W_{ss} = 80$

The time evolution of the rabbit and wolf populations is shown in figure 1.

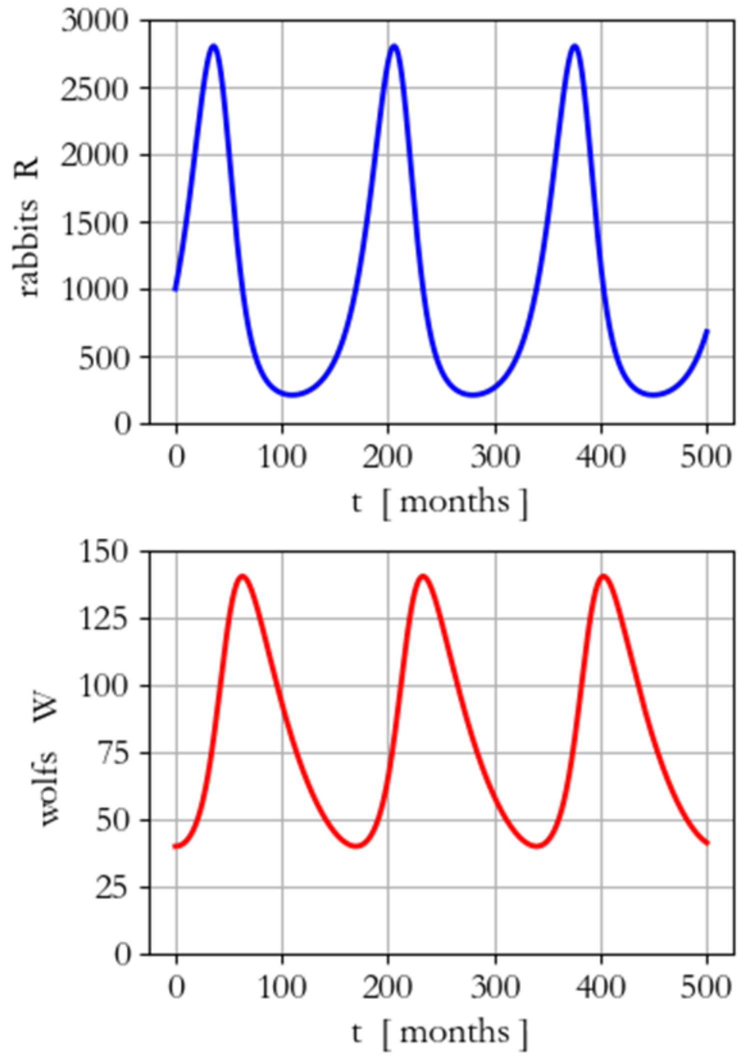

Fig. 1A. Time evolution of the predator-prey system.

The two population graphs can be plotted in the one window as shown in figure 1B.

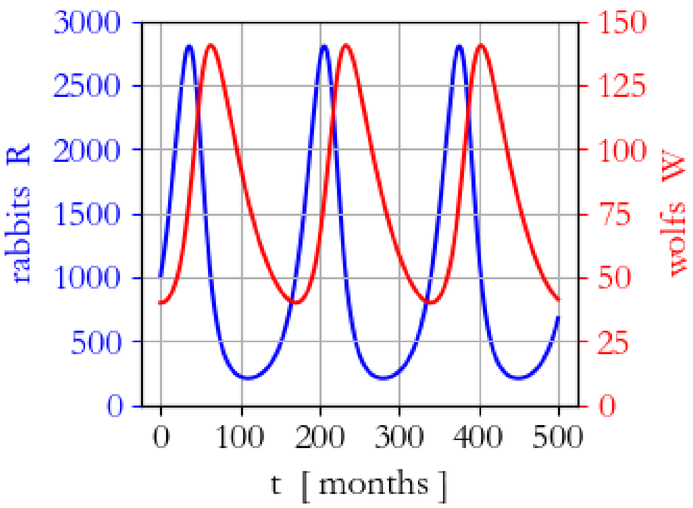

Fig. 1B. Time evolution of the predator-prey system.

The two populations oscillate with the same period of 170 months, with a peak in the rabbit population occurring earlier then the peak in the wolf population by 27 months (0.16 of the cycle time).

If you run the simulation (cs_003.py) with the initial conditions $\left(R(0) = R_{ss} = 1000, W(0) = W_{ss} = 80\right)$ then the populations do not change. This means that rabbits are just enough to support a constant wolf population. There are neither too many wolves (which would result in fewer rabbits) nor too few wolves (which would result in more rabbits)

PHASE SPACE

An illuminating way to study the dynamics of multiple variable systems is to use a phase space diagram.

A **phase space** of a dynamical system is a theoretical space where every state of the system is mapped to a unique spatial location.

Phase space diagrams

- Will show happen to a system's state in the long run. For a deterministic dynamical system, its future state is uniquely determined by its current state. Trajectories will never branch off in phase space, once you specify an initial state of the system, the trajectory is uniquely determined. Trajectories may diverge to infinity, converge to a point, or remain dynamically changing yet stay in a confined region. A converging point or a confined region is called an **attractor**. For each attractor, you can find the set of all the initial states from which you will eventually end converging into that attractor (the basin of attraction). When there is more than one the phase space can be divided into several different regions. Such a "map" drawn on the phase space reveals how sensitive the system is to its initial conditions. If one region is dominating in the phase space, the system's fate doesn't depend much on its initial condition. But if there are several regions that are equally represented in the phase space, the system's fate sensitively depends on its initial condition.

- The stability of the system's states can be visualised from the trajectories in a phase space diagram. Convergence to an attractor means the system's state is stable in that area. If trajectories are diverging from a certain area, then the system's state is unstable in that area.

The phase portrait (phase space diagram) for figure 1 is shown in figure 2. The red dot shows the location of the equilibrium point (1000, 80) and the green dot shows the initial location (1000, 40).

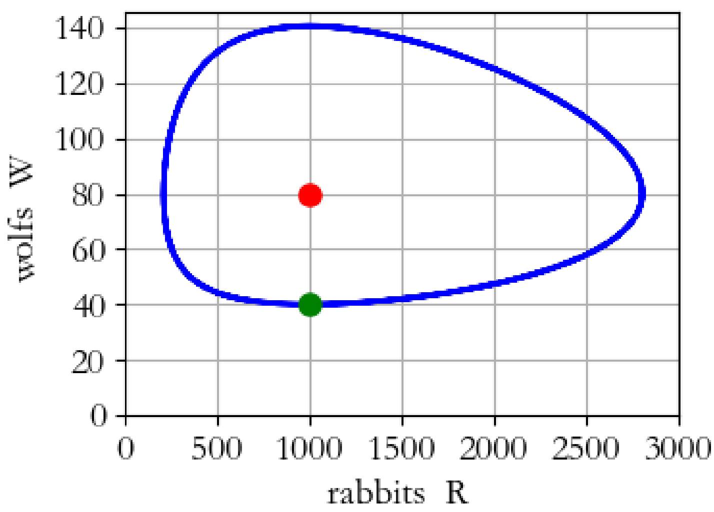

Fig. 2. Phase portrait (phase space diagram): Initial conditions (R = 1000, W = 40) shown by green dot and the fixed-point (R_{ss} = 1000, W_{ss} = 80) by red dot.

At a maximum or minimum in either the rabbit or wolf population, the rates of change of a population must be zero (dR/dt = 0 or dW/dt = 0). Therefore, at a maximum or minimum one of the populations will be equal to its steady-state value:

$$R_{max} = 2803 \quad R_{min} = 210 \quad W = W_{ss} = 80$$

$$W_{max} = 140 \quad W_{min} = 40 \quad R = R_{ss} = 1000$$

At the time of a maximum in the rabbit population, the wolf population is increasing. So, in the phase space trajectory, the direction of the orbit is anticlockwise as shown in figure 3 by the direction field plot and in figure 4 by the direction of the streamline trajectories.

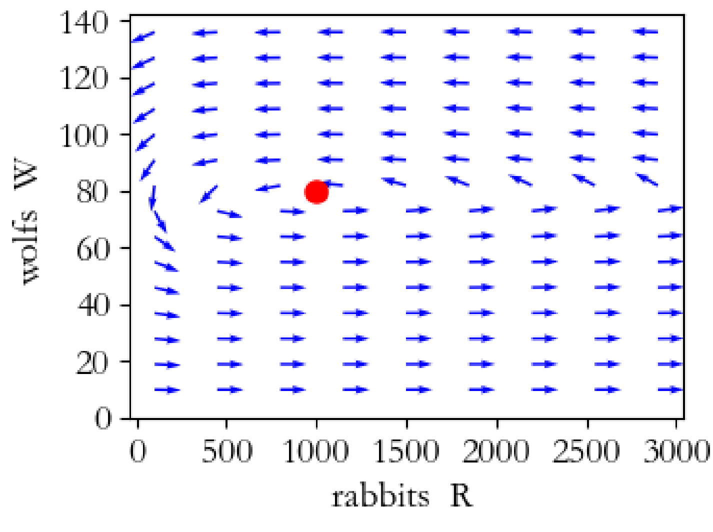

Fig. 3. Direction field for the predator-prey system. All trajectories will rotate anticlockwise about the fix-point attractor (1000, 80).

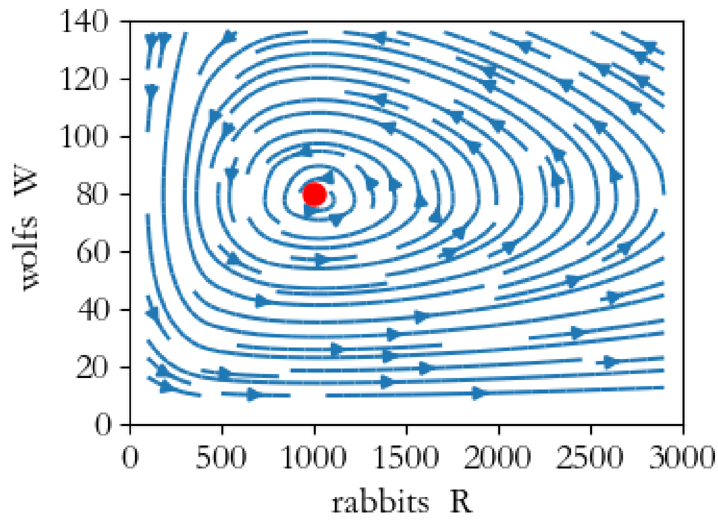

Fig. 4. Streamline plot for the predator-prey system. All trajectories will rotate about the fix-point attractor (1000, 80) in an anticlockwise direction.

For the initial condition (1000, 40) there aren't enough wolves to maintain a balance between the two populations leading to an increase in the rabbit population. Hence, there will an increasing number of wolfs as the rabbit population increases to a maximum value. When there are many wolfs, the rabbits have a harder time avoiding them resulting in a decline in rabbit numbers as the number of wolfs further increases. Now, the rabbit numbers are declining, this means that at some later time the wolf population starts to fall and this is beneficial to the rabbits. So, the rabbit population later starts to increase again. As a consequence, the wolf population eventually starts to increase as well. When the populations return to their initial values the entire cycle begins again.

Figure 5 shows the multiple trajectories in phase space for different initial conditions (green dots). In all cases, the trajectory is an orbit about the fixed-point (red dot). The tighter the orbit than the smaller the cycle time (period).

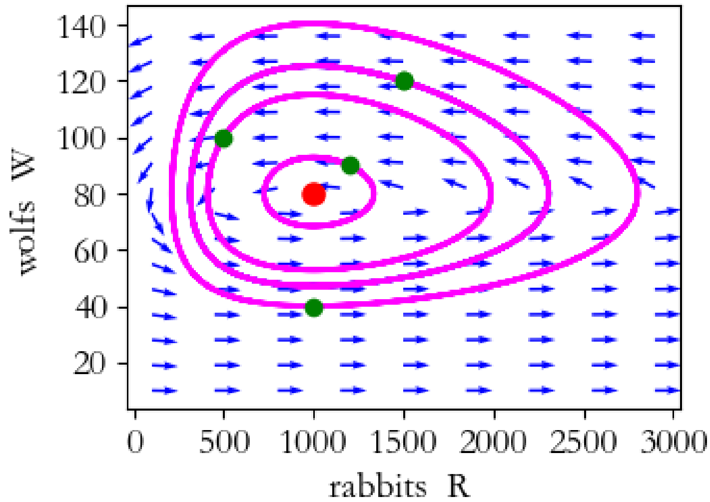

Fig. 5. Phase portrait of the system for multiple initial conditions.

REFERENCES

LibreTexts Mathematics

https://www.math.stonybrook.edu/~azinger/mat127-spr22/PredatorPreySystems.pdf