DOING PHYSICS WITH PYTHON

DYNAMICAL SYSTEMS

[2D] LINEAR DYNAMICAL SYSTEMS

Theoretical considerations

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DOWNLOAD DIRECTORIES FOR PYTHON CODE

[Google drive]

[GitHub]

cs200.py ds1421.py ds1422.py ds1423.py

The Python Codes solve a pair of linear ODEs in x and y. The solution gives the time evolution of the two variables and the phase portrait (quiver plot and streamplot) using nullclines, vector fields, and eigenvectors. One can find and classify critical points in the phase plane.

The letters used for physical quantities are different from those used in Jason's video. Greek letters are avoided and letters used in this paper are closely related to the letters used in the Python Code.

INTRODUCTION

This article considers how Python can be used to solve [2D] linear dynamical systems. The [2D] systems are described by a pair of ordinary differential equations (ODEs) in *x* and *y*. The ODEs are solved numerically using the Python function **odeint**. The solutions for *x* and *y* are displayed graphically as time evolution plots and phase portrait plots. For a linear system, the ODEs can be expressed as

$$\dot{x} = a_{00}\,x + a_{01}\,y \qquad \dot{y} = a_{10}\,x + a_{11}\,y$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

The phase portrait is a [2D] figure showing how the qualitative behaviour of system is determined as *x* and *y* vary with *t*. With the appropriate number of trajectories plotted, it should be possible to determine where any trajectory will end up from any given initial condition.

The **vector field** gives the gradients *dy* and *dx*. The slope of the trajectories at each point in the vector field is given by

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$$

The contour lines for which *dy/dx* is a constant are called **isoclines**. The contour lines for which $dx/dt = 0$ and $dy/dt = 0$ are called **nullclines**.

Isoclines may be used to help with the construction of the phase portrait. For example, the nullclines for which $\dot{x} = 0$ and $\dot{y} = 0$ are used to determine where the trajectories have vertical and horizontal tangent lines, respectively. If $\dot{x} = 0$, then there is no motion horizontally, and trajectories are either stationary or move vertically. When $\dot{y} = 0$, then there is no motion vertically, and trajectories are either stationary or move horizontally.

Systems in which $\det(\mathbf{A}) \neq 0$ is **simple**, and the Origin is then the only critical point.

A linear system is **non-simple** if the matrix **A** is singular, i.e., det(**A**) = 0, and at least one of the eigenvalues is zero then this system has critical points other than the Origin (multiple fixed points].

An **equilibrium** occurs at **critical points** of a dynamical system generated by system of ordinary differential equations (ODEs) where a solution that does not change with time. Let (x_e, y_e) be the fixed point (equilibrium, critical or steady-state point). To find the fixed point, you only need to solve the pair of equations governing the system.

$$\dot{x}(x_e) = 0 \quad \dot{y}(y_e) = 0 \;\Rightarrow\; a_{00}\, x_e + a_{01}\, y_e = 0 \qquad a_{10}\, x_e + a_{11}\, y_e = 0$$

Nullclines are curves or lines within a phase plane of a system of differential equations, defined by the condition that one of the rates of change is zero. For a system of two equations, the "x-nullcline" is where $dx/dt = 0$, causing vectors to move purely vertically, while the "y-nullcline" is where $dy/dt = 0$, causing vectors to move purely horizontally. By finding these nullclines, one can sketch a phase portrait to understand how the system's solutions behave over time, with intersections of nullclines representing fixed points.

- Along an x-nullcline: $dx/dt = 0$, so the solution's state changes only in the y-direction (up or down).
- Along a y-nullcline: $dy/dt = 0$, so the solution's state changes only in the x-direction (left or right).
- In each region, the signs of dx/dt and dy/dt indicate the overall direction of flow.
- The intersections of the nullclines are the equilibrium points, where both dx/dt and dy/dt are simultaneously zero.

Nullclines provide a qualitative understanding of the system's dynamics without needing to find exact analytical solutions, which can be difficult for nonlinear systems. They help predict the overall behaviour of solutions, such as whether they will approach equilibrium points, oscillate, or move in specific directions and the location of fixed points.

The [2D] planar system of linear ODEs are

$$\dot{x} = a_{00}\,x + a_{01}\,y \qquad \dot{y} = a_{10}\,x + a_{11}\,y$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$$

and analytical solutions are given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_0 \begin{pmatrix} f_{00} \\ f_{10} \end{pmatrix} e^{L_0 t} + c_1 \begin{pmatrix} f_{10} \\ f_{11} \end{pmatrix} e^{L_1 t}$$

$$x(t) = c_0\, f_{00}\, e^{L_0 t} + c_1\, f_{10}\, e^{L_1 t}$$

$$y(t) = c_0\, f_{10}\, e^{L_0 t} + c_1\, f_{11}\, e^{L_1 t}$$

where L_0 and L_1 are the **eigenvalues** $\left(L \equiv \lambda\right)$ of the matrix **A**

and **F** is the eigenvector matrix

$$\mathbf{F} = \begin{pmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{pmatrix} \qquad \mathbf{F}_0 = \begin{pmatrix} f_{00} \\ f_{10} \end{pmatrix} \qquad \mathbf{F}_1 = \begin{pmatrix} f_{01} \\ f_{11} \end{pmatrix}$$

The eigenvalues and eigenvector matrix are found using the Python function **eig(A)**. The *c* coefficients are found by solving the pair of equation using the Python function **solve**.

The *x*,*y* solution depends upon the exponential values of the eigenfunctions which may be complex with real and imaginary components. So, we only have to consider the exponential function of the form.

Let an eigenvalue be expressed at $L = a + b\,j$ and the exponential terms as

$$\exp(Lt) = \exp((a+bj)t) = \exp(at)\exp(btj)$$
$$\exp(Lt) = \exp(at)(\cos(bt) + \sin(bt)j)$$
$$\operatorname{Re}[\exp(Lt)] = \exp(at)\cos(bt)$$

You can now see clearly, the real and imaginary parts of the eigenvalues determine the behaviour of the flow. There is the exponential growth or exponential decay term $\exp(at)$ and the oscillatory term $\cos(bt)$. For the flow, the Origin (0, 0) is always a fixed point of the system.

- Complex eigenvalue $b \neq 0 \;\; a = 0 \Rightarrow$ oscillatory flow (SHM motion) which leads to closed elliptical orbits in the phase portrait plot.
- Complex eigenvalue $b \neq 0 \; a > 0 \Rightarrow$ oscillatory flow with increasing amplitude.
- Complex eigenvalue $b \neq 0 \; a < 0 \Rightarrow$ oscillatory flow with increasing amplitude.
- Real eigenvalue $b = 0$, $a > 0 \Rightarrow$ flow is away from the fixed point at the Origin (0, 0).
- Real eigenvalue $b = 0$, $a < 0 \Rightarrow$ flow is towards the fixed point at the Origin (0, 0).

The stability of typical equilibria of smooth ODEs is determined by the sign of real part of the eigenvalues of the Jacobian matrix **J**. In [2D] systems the Jacobian matrix **J** is

$$\mathbf{J}(x,y) = \begin{pmatrix} \partial f / \partial x & \partial f / \partial y \\ \partial g / \partial x & \partial g / \partial y \end{pmatrix}$$

and for the planar linear system

$$\mathbf{J}(x,y) = \begin{pmatrix} \partial f / \partial x & \partial f / \partial y \\ \partial g / \partial x & \partial g / \partial y \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \qquad \mathbf{J} \equiv \mathbf{A}$$

These eigenvalues are often referred to as the eigenvalues of the equilibrium which are either both real or a complex-conjugate. The eigenvalues and eigenfunctions can be found using the Python function **eig**. $(\lambda \equiv L)$

Eigenvalues λ_1, λ_2	Stability of critical point (equilibrium or fixed points)
distinct, real, and positive $\lambda_1 \neq \lambda_2 \quad \lambda_1 > 0 \quad \lambda_2 > 0$	Unstable node
Both eigenvalues zero $\lambda_1 = \lambda_2 = 0$	Unstable
distinct, real, and negative $\lambda_1 \neq \lambda_2 \quad \lambda_1 < 0 \quad \lambda_2 < 0$	Stable node
One eigenvalue is positive and the other negative $\lambda_1 > 0 \quad \lambda_2 < 0$	Saddle point
Repeated eigenvalues $\lambda_1 = \lambda_2 > 0$	Unstable If there are two linearly independent eigenvectors, then the critical point is called a singular node. If there is one linearly independent eigenvector, then the critical point is called a degenerate node.

Repeated eigenvalues $\lambda_1 = \lambda_2 < 0$	Stable
Complex eigenvalues $\lambda = a + b\,j \quad b \neq 0$	
$a > 0$	Unstable oscillator: amplitude grows with time
$a = 0$	Stable: undamped oscillator
$a < 0$	Stable: damped oscillator

The eigenvectors give the **manifolds** of the system and the manifolds maybe stable or unstable. For the eigenvector (x_J, y_J), the manifold is given by the line from the Origin (0, 0) through the point (x_J, y_J).

The manifolds can also be determined from the analytical solutions of the ODEs

$$x(t) = c_0\, f_{00}\, e^{L_0 t} + c_1\, f_{10}\, e^{L_1 t}$$
$$y(t) = c_0\, f_{10}\, e^{L_0 t} + c_1\, f_{11}\, e^{L_1 t}$$

$c_0 = 0$ manifold is the straight line from the Origin through the point (x_{c0}, y_{c0}) evaluated at some large time t.

$$x_{c0} = c_1\, f_{10}\, e^{L_1 t} \qquad y_{c0} = c_1\, f_{11}\, e^{L_1 t}$$

$c_1 = 0$ manifold is the straight line from the Origin through the point (x_{c1}, y_{c1}) evaluated at some large time t.

$$x_{c1} = c_0\, f_{10}\, e^{L_1 t} \quad y_{c0} = c_0\, f_{11}\, e^{L_1 t}$$

We can consider the two straight lines when $c_0 = 0$ and $c_1 = 0$ and the two eigenvalues are real and of opposite signs, $L_0 > 0$ and $L_1 < 0$.

$c_1 = 0$ and $L_0 > 0$

$$x(t) = c_0\, f_{00}\, e^{L_0 t} \quad y(t) = c_0\, f_{10}\, e^{L_0 t} \Rightarrow$$

$x(t)$ and $y(t)$ increase exponentially with time t

$t \to \infty \quad \left|x(t), y(t)\right| \to (\infty, \infty)$

The flow is directed along this line away from the saddle point (0, 0)

Unstable – everything is expanding

$c_0 = 0$ and $L_1 < 0$

$$x(t) = c_1\, f_{10}\, e^{L_1 t} \quad y(t) = c_1\, f_{11}\, e^{L_1 t}$$

$x(t)$ and $y(t)$ decrease exponentially with time t

$t \to \infty \quad \left|x(t), y(t)\right| \to (0, 0)$

The flow is directed along this line towards the saddle point (0, 0)

Stable – everything is pulled into the saddle point.