DOING PHYSICS WITH PYTHON

[2D] DYNAMICAL SYSTEMS

LINEAR PLANAR SYSTEMS

MULTIPLE FIXED POINTS

Ian Cooper
matlabvisualphysics@gmail.com

DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive

GitHub

ds1421.py

Jason Bramburger
Linear Planar Systems - Dynamical Systems | Lecture 14

https://www.youtube.com/watch?v=b8eJb5uwNZI

Greek letters are avoided and letters used in this paper are closely related to the letters used in the Python Code.

This paper considers [2D] linear dynamical systems which have multiple fixed points. The Jacobian matrix **J** has real eigenvalues one of which is zero.

$$\lambda_1 \neq 0 \ \text{ real and } \lambda_0 = 0$$

SIMULATIONS

Example 1 Unstable multiple critical points $\lambda_1 > 0 \quad \lambda_2 = 0$

System: $\dot{x} = x \qquad \dot{y} = 0$

Dynamics:

$$f = x \quad df/dx = 1 \quad df/dy = 0$$
$$g = 0 \quad dg/dx = 0 \quad dg/dy = 0$$
$$\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \lambda_1 = 1 \quad \lambda_2 = 0$$
$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \det(\mathbf{A}) = 0$$

Initial conditions (x0, y0)

(1.00, 1.00) (-1.00, -1.00) (-2.00, -2.00) (-1.50, -1.50)

(0.00, 0.00) (2.00, 2.00)

A matrix: a11 = 1.0 a12 = 0.0 a11 = 0.0 a12 = 0.0

Determinant A = 0.00

Eigenvalues Jacobian J = 1.00 0.00

This linear system is **non-simple** since the matrix **A** is singular, det(**A**) = 0, and at one of the eigenvalues is zero. Therefore, this system has critical points other than the Origin (multiple fixed points).

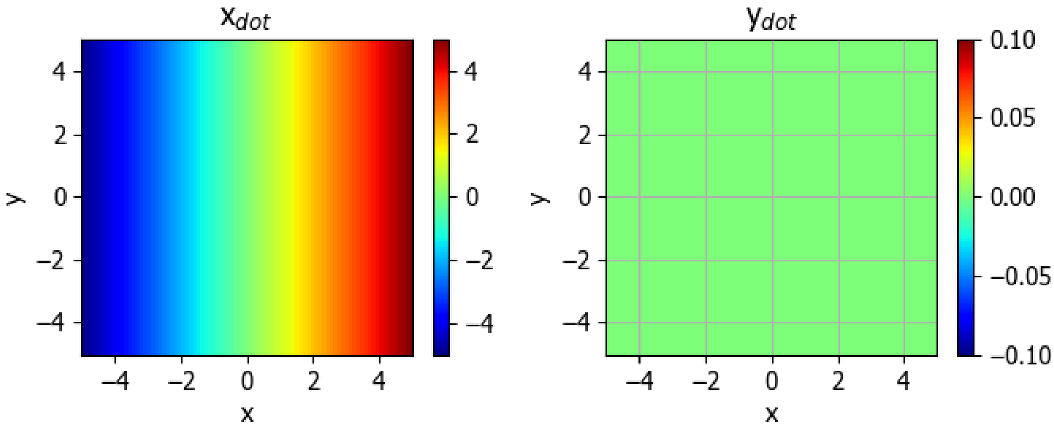

Fig. 1.1. [2D] view of the system equations. The flow is horizontal away from the $x = 0$.

$$x(0) < 0 \;\; y(0) \neq 0 \Rightarrow t \to \infty \;\;\; x \to -\infty \;\;\; y = y(0)$$
$$x(0) > 0 \;\; y(0) \neq 0 \Rightarrow t \to \infty \;\;\; x \to +\infty \;\;\; y = y(0)$$

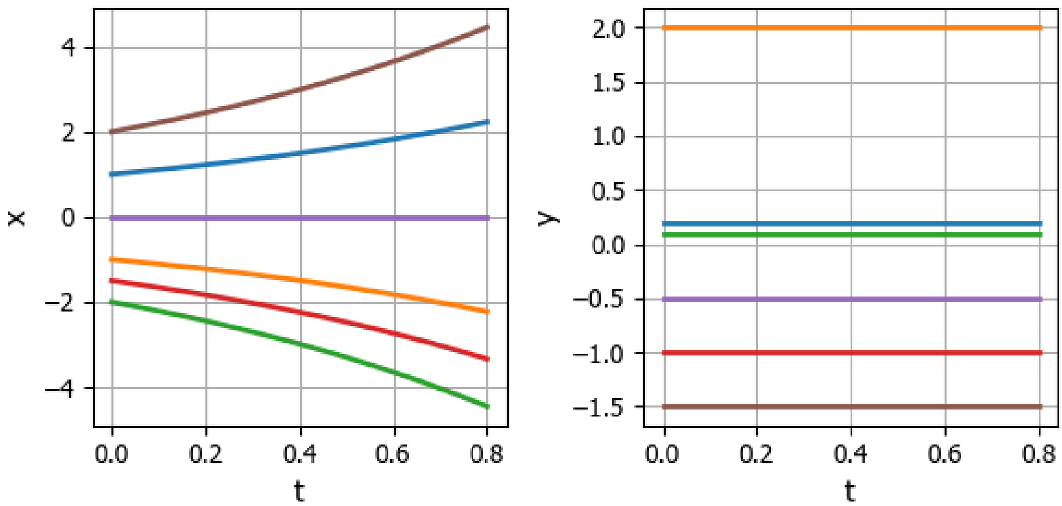

Fig. 1.2. Time evolution of the x and y trajectories.

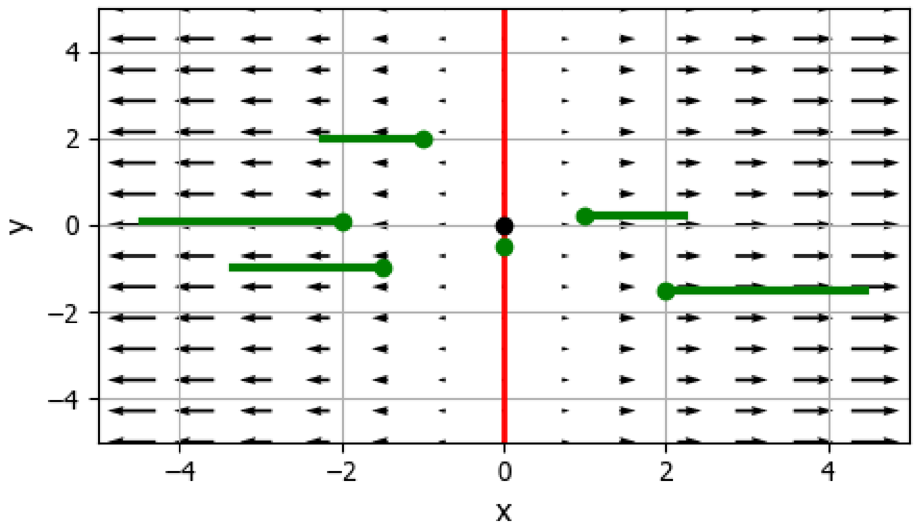

Fig. 1.3. Vector field: quiver plot. There are an infinite number of critical points lying along the ***y*-axis**. Six trajectories (orbits) in phase space for a time interval $\Delta t = 0.40$. The **black dot** is the equilibrium point of the system at the Origin (0,0), and the **green dots** are for the different initial conditions ($x(0), y(0)$). The length of the trajectory is shorter when the initial conditions are nearer the equilibrium point at the Origin. Solutions of the ODEs are $x(t)$ and $y(t)$.

$y(0) \neq 0 \quad t \to \infty \quad x(t) \to \pm\infty \quad y(t) = y(0)$ where $x(t)$ changes exponentially.

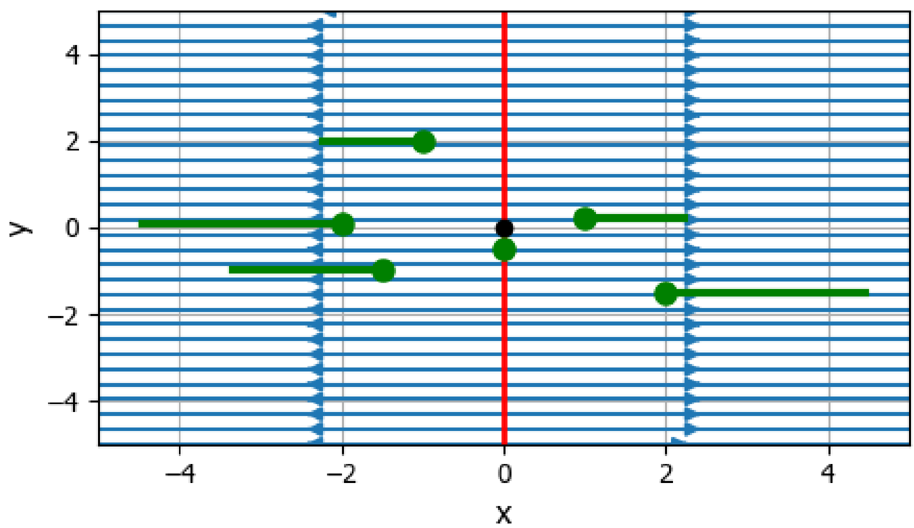

Fig. 1.5. Vector field: streamplot. There are an infinite number of critical points lying along the ***y*-axis**.

The determinant is zero $\det(\mathbf{A}) = 0$ and the eigenvalues are $\lambda_0 = 1$ and $\lambda_1 = 0$. One eigenvalue is real and positive and the other is zero. Therefore, the equilibrium points on the y-axis are **unstable**.

The critical points are found by solving the equations

$$\dot{x} = 0 \qquad \dot{y} = 0$$

which has the solution $x = 0$, y = constant. Thus, there are an infinite number of critical points lying along the y-axis. The direction field has gradient given by

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = 0 \qquad x \neq 0$$

This implies that the direction field is horizontal for points not

on the y-axis. The direction vectors may be determined from the equation $\dot{x} = x$ since if $x > 0$, then $\dot{x} > 0$, and the trajectories move in +x direction and if $x < 0$, then $\dot{x} < 0$, and trajectories move in the -x direction.

Example 2 Points along the y-axis are stable nodes

$$\lambda_1 < 0 \quad \lambda_2 = 0$$

System: $\dot{x} = -x \qquad \dot{y} = 0$

Dynamics:

$$f = x \quad df / dx = -1 \quad df / dy = 0$$

$$g = 0 \quad dg / dx = 0 \quad dg / dy = 0$$

$$\mathbf{J} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \quad \lambda_1 = -1 \quad \lambda_2 = 0$$

$$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \quad \det(\mathbf{A}) = 0$$

Initial conditions (x0, y0)

(1.00, 1.00) (-1.00, -1.00) (-2.00, -2.00) (-1.50, -1.50)

(0.00, 0.00) (2.00, 2.00)

A matrix: a11 = -1.0 a12 = 0.0 a11 = 0.0 a12 = 0.0

Determinant A = 0.00

Eigenvalues Jacobian J = -1.00 0.00

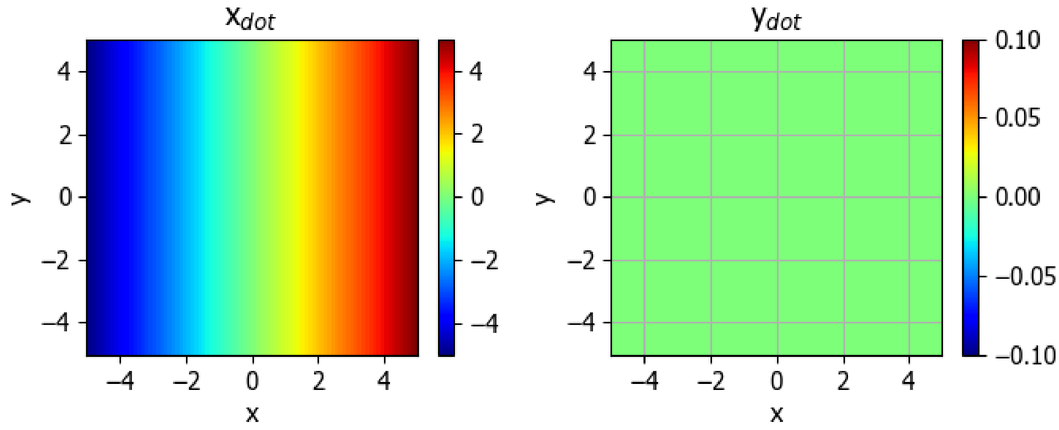

Fig. 2.1. [2D] view of the system equations.

$$x(0) < 0 \Rightarrow t \to \infty \quad x \to 0 \quad y = y(0)$$
$$x(0) > 0 \Rightarrow t \to \infty \quad x \to 0 \quad y = y(0)$$

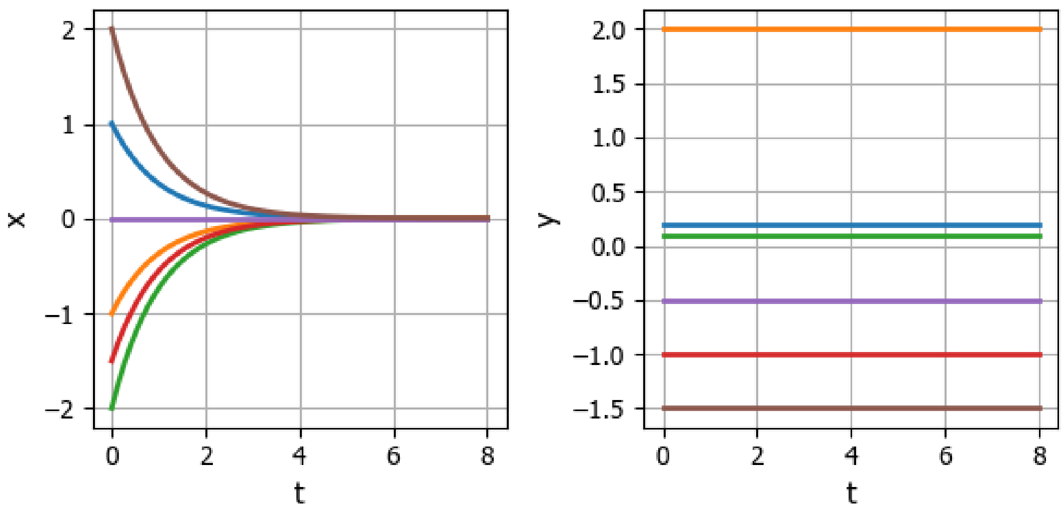

Fig. 2.2. Time evolution of the x and y parameters.

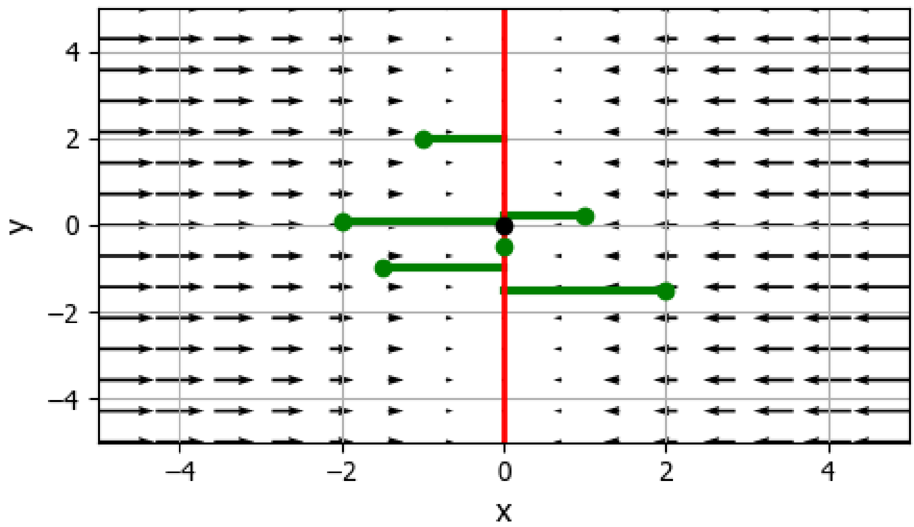

Fig. 2.4. Vector field: quiver plot. There are an infinite number of critical points lying along the ***y*-axis**.

Six trajectories (orbits) in phase space for a time interval $\Delta t = 0.40$. The **black dot** is the equilibrium point at the Origin (0,0), and the **green dots** are for the different initial conditions ($x(0)$, $y(0)$). Solutions of the ODEs are $x(t)$ and $y(t)$. For all initial conditions:

$t \to \infty \quad x(t) \to 0 \quad y(t) = y(0)$.

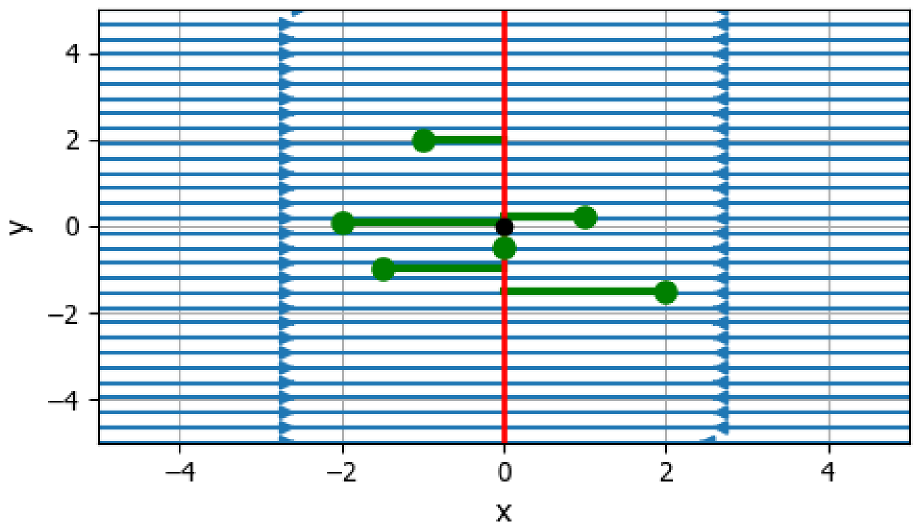

Fig. 2.5. Vector field: streamplot.

The determinant is zero $\det(\mathbf{A}) = 0$ and the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 0$. One eigenvalue is real and negative and the other is zero. Therefore, there is an infinite number of equilibrium points along the y-axis ($x = 0$). All trajectories, head towards $x = 0$ with constant y.