DOING PHYSICS WITH PYTHON

DYNAMICAL SYSTEMS

LINEAR PLANAR [2D] SYSTEMS

REAL NON-ZERO EIGENVALUES

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DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive

GitHub

cs200.py ds1402.py ds1422.py

Values for the initial conditions u_0, the eigenvalues L, eigenvectors **F** and c coefficients are displayed in the Console Window:

Jason Bramburger
Linear Planar Systems - Dynamical Systems | Lecture 14

https://www.youtube.com/watch?v=b8eJb5uwNZI

Greek letters are often avoided and letters used in this paper are closely related to the letters used in the Python Code.

This paper considers [2D] linear dynamical systems which have real non-zero eigenvalues.

SIMULATIONS

Example 1   ds1422.py

real positive eigenvalues $\lambda_0 > 0 \quad \lambda_1 > 0$

$$\lambda > 0 \quad t \to \infty \quad \exp(\lambda t) \quad \to \infty \Rightarrow$$

UNSTABLE fixed point at Origin (0, 0)

System: $\dot{x} = 2x + y \qquad \dot{y} = x + 2y$

A matrix: a00 = 2.00 a01 = 1.00 a10 = 1.00   a11 = 2.00

Determinant A = 3.00000

Initial conditions (x0, y0)

(-1.00, 2.00)   (-2.90, 1.31)   (-1.50, -0.80)   (1.00, -2.00)

(3.00, -1.50)   (1.50, -0.50)   (0.10, 0.10)

Eigenvalues Jacobian J1 = 3.00 J2 = 1.00

Eigenfunction Jacobian e0 = 1.00 1.00

Eigenfunctions Jacobian e1 = -1.00 1.00

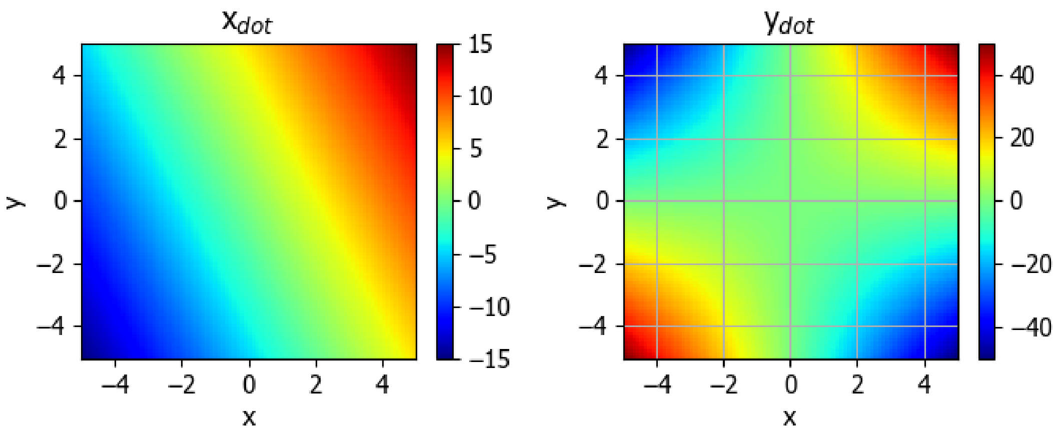

Fig. 1.1.   [2D] view of the system equations.

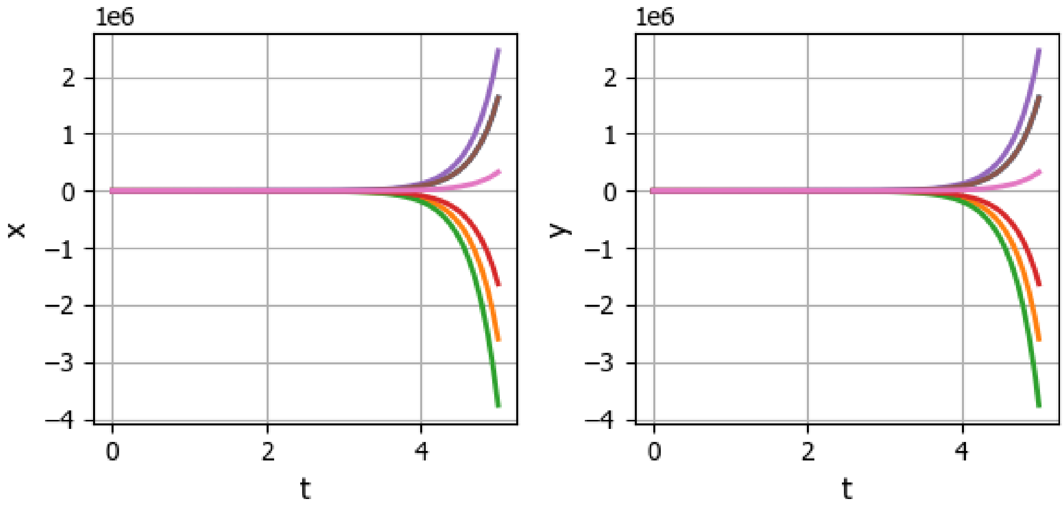

Fig. 1.2. Time evolution of the x and y parameters. All trajectories diverge to either $+\infty$ or $-\infty$.

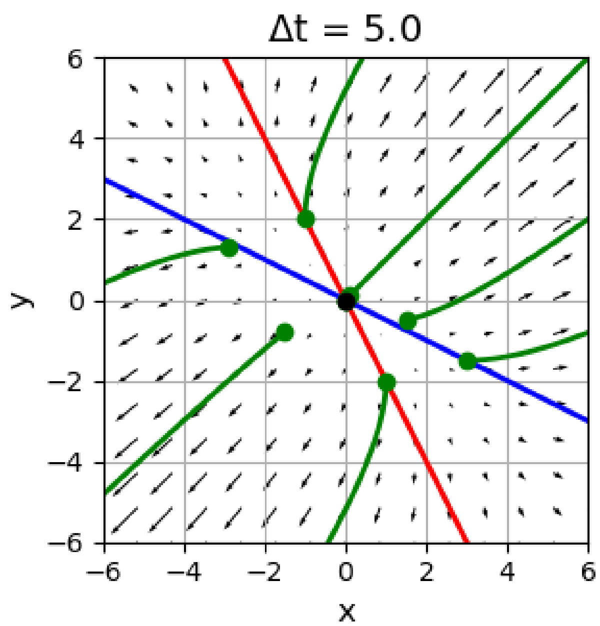

Fig. 1.3. Vector field: quiver plot.

Red: x-nullcline $(\dot{x}=0)$ Blue: y-nullcline $(\dot{y}=0)$

green: initial conditions (x_0, y_0) and trajectories

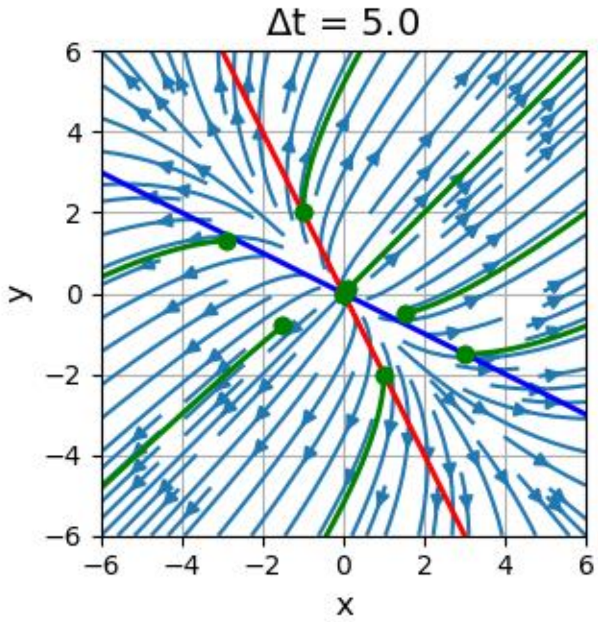

Fig. 1.4. Vector field: streamplot.

The intersection of the *x*-nullcline and the *y*-nullcline gives the **equilibrium point** (0,0). The determinant is non-zero $\det(\mathbf{A}) = 3 \neq 0$ and the eigenvalues are $\lambda_0 = 1$ and $\lambda_1 = 3$. Both eigenvalues are real and positive. Therefore, the equilibrium point at the Origin (0,0) is an **unstable node**.

Eigenfunctions and manifolds

In 2x2 systems, eigenvectors play a crucial role in understanding and simplifying the system's behaviour. They are used to define **invariant manifolds**, which are surfaces in the state space where the system's trajectories remain confined and trajectories starting on the manifold remain on it for all time. The **eigenvalues** and **eigenvectors** (**eigenfunctions**) reveal the system's stability and the direction of its trajectories and gives information about the local behaviour around fixed points as shown in figure 1.5

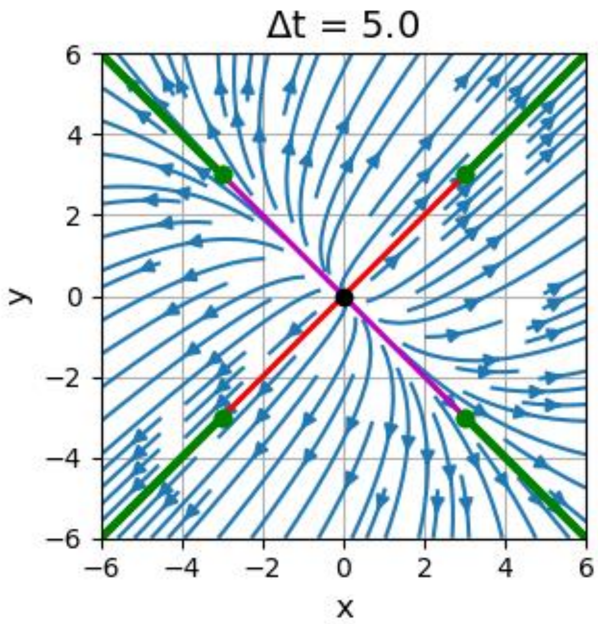

Fig. 3.5. The **manifolds** are defined by the eigenvectors **e0 (1,1)** and **e1 (-1,1)**. Both manifolds are **unstable**. One passes through (0, 0) and (1, 1) and the other through (0, 0) and (1, −1). All trajectories lying on the unstable manifolds diverge to either $+\infty$ or $-\infty$. Trajectories starting on a manifold stay on the manifold.

Example 2   ds1422.py

SADDLE NODE

One eigenvalue is **real** and **positive** and the other is **real** and **negative**, the critical point at the Origin is a **saddle point**.

$$\begin{aligned} &\lambda_0 < 0 \quad t \to \infty \quad \exp(\lambda_0 t) \to 0 \\ &\lambda_1 > 0 \quad t \to \infty \quad \exp(\lambda_1 t) \to \infty \end{aligned}$$

saddle node (unstable) at (0, 0)

System: $\dot{x} = x + y \quad \dot{y} = 4x - 2y$

$$a_{00} = 1 \;\; a_{01} = 1 \;\; a_{10} = 4 \;\; a_{11} = -2$$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$$

A matrix: a00 = 1.00 a01 = 1.00 a10 = 4.00   a11 = -2.00

Determinant A = -6.00000

Initial conditions (x0, y0)

(-4.0000, 5.0000)   (-3.0000, 5.0000)   (-2.0000, 5.0000)

(-1.0000, 5.0000)   (-1.5000, 5.0000)   (0.0000, 5.0000)

(1.0000, -5.0000)   (2.0000, -5.0000)   (3.0000, -5.0000)

(4.0000, -5.0000)

Eigenvalues   2.000   -3.000

Eigenfunctions [1.   -0.25]   [1. 1.]

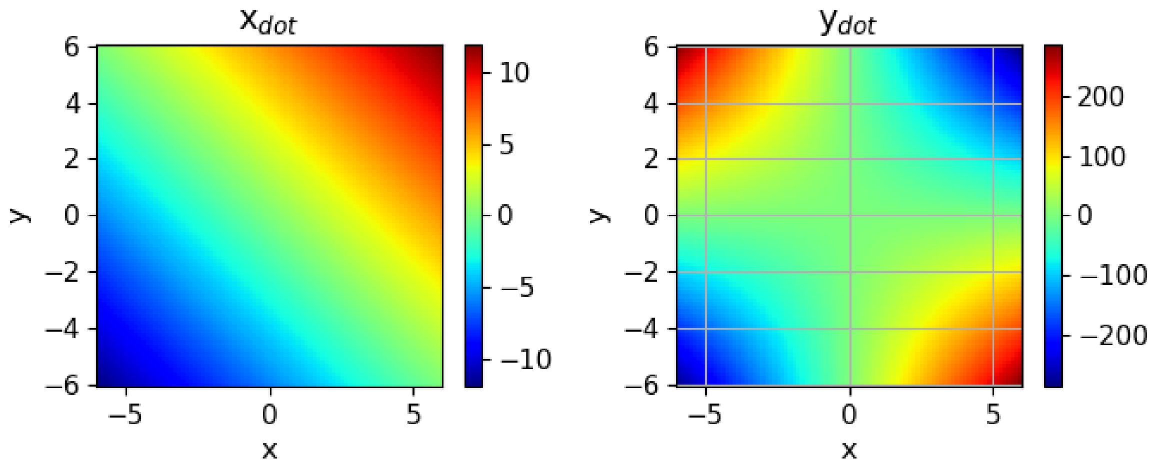

Fig. 2.1. [2D] view of the system equations.

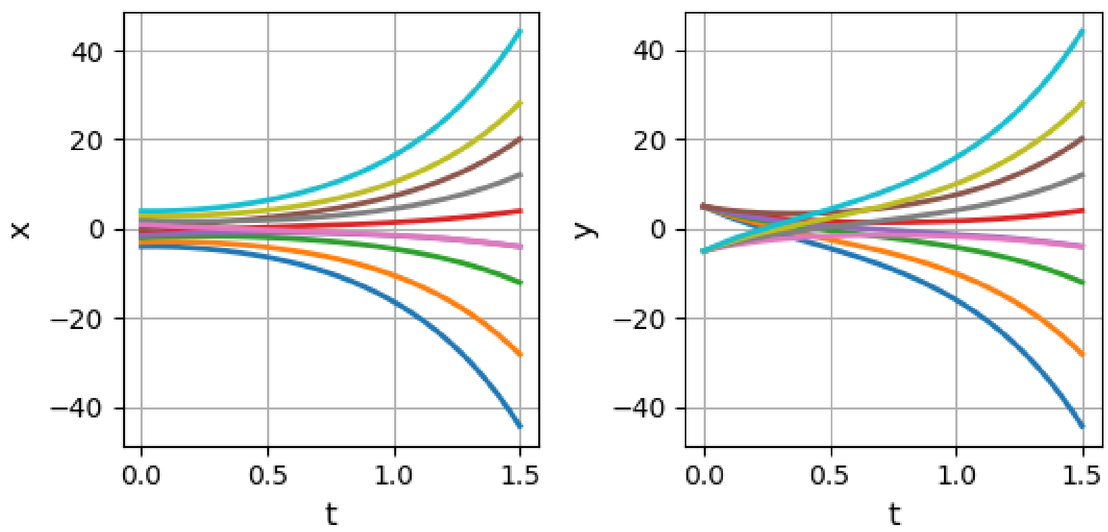

Fig. 2.2. Time evolution of the x and y trajectories.

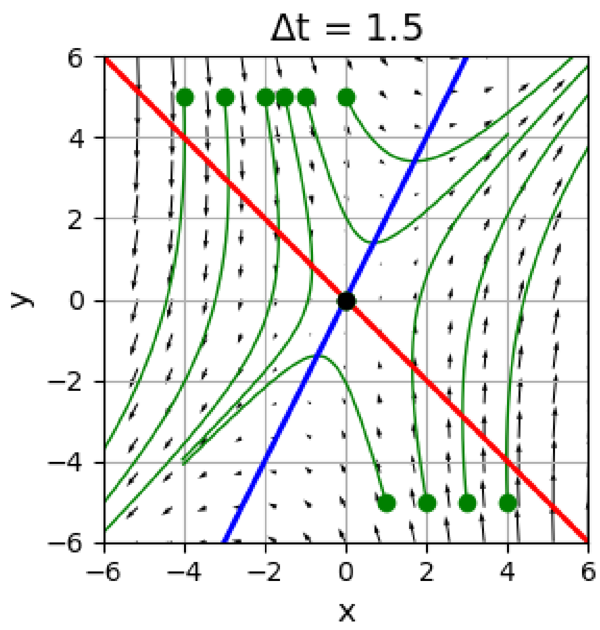

Fig. 2.3. Vector field: quiver plot.

Red: x-nullcline $(\dot{x}=0)$ Blue: y-nullcline $(\dot{y}=0)$

green: initial conditions (x_0, y_0) and trajectories

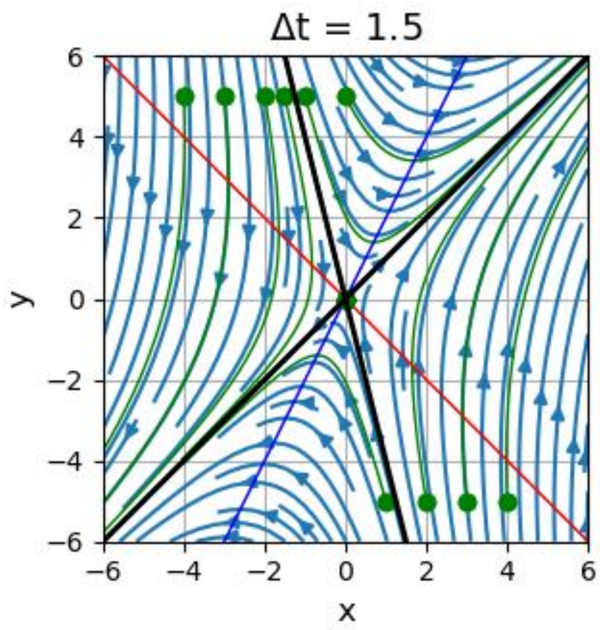

Fig. 2.4. Vector field: streamplot. The **manifolds** are defined by the eigenvectors. Stable **e1 (1,1)**. and unstable **e0 (2,1)** manifolds (**black**).

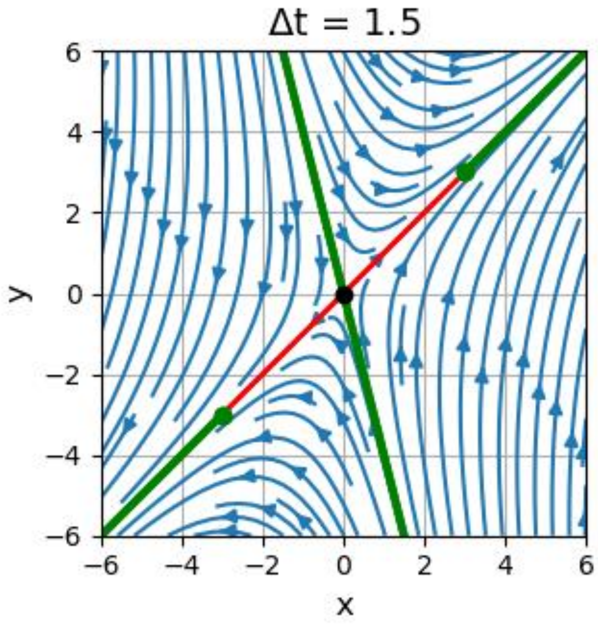

Fig. 2.5. If the initial conditions are on a manifold, then the trajectory follows the manifold towards the Origin for the stable manifold and away from the Origin for the unstable manifold. Trajectories lying on the stable manifold tend to the Origin as $t \to \infty$ but never reach it.

Example 3 ds1402.py SADDLE NODE

System: $\dot{x} = x + y \quad \dot{y} = 4x - 2y$ (same as Example 2)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_0 \begin{pmatrix} f_{00} \\ f_{10} \end{pmatrix} e^{L_0 t} + c_1 \begin{pmatrix} f_{10} \\ f_{11} \end{pmatrix} e^{L_1 t}$$

$$x(t) = c_0\, f_{00}\, e^{L_0 t} + c_1\, f_{10}\, e^{L_1 t} \qquad y(t) = c_0\, f_{10}\, e^{L_0 t} + c_1\, f_{11}\, e^{L_1 t}$$

$$t \to \infty \quad e^{-3t} \to 0 \quad \left| e^{+2t} \right| \to \infty$$

When one eigenvalue is real and positive and the other eigenvalue is real and negative, then there is a **saddle point**

Real eigenvalues: $L_0 > 0$ and $L_1 < 0 \Rightarrow$ **saddle point**

Initial conditions [-1.8 5.5]

Eigenvalues [2. -3.] Eigenvectors [[0.71 -0.24] [0.71 0.97]]

coeff c = [-0.481 6.02]

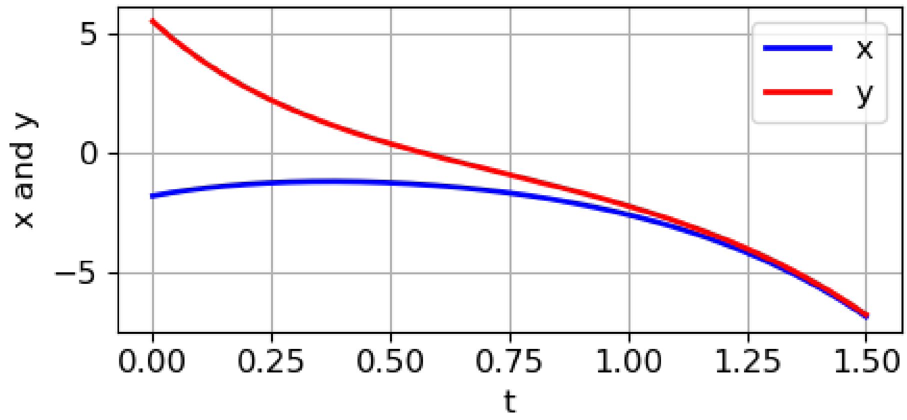

Fig. 3.1. Time evolution for the flow of x and y. The flow is attracted to the c_1 straight line (figure 3).

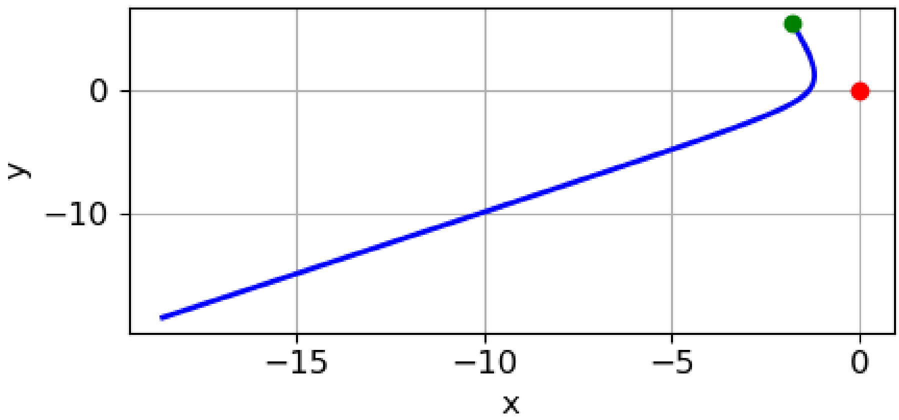

Fig. 3.2. Initially the flow is attracted to the saddle point (0, 0) and then repelled along the c_1 straight line (figure 3).

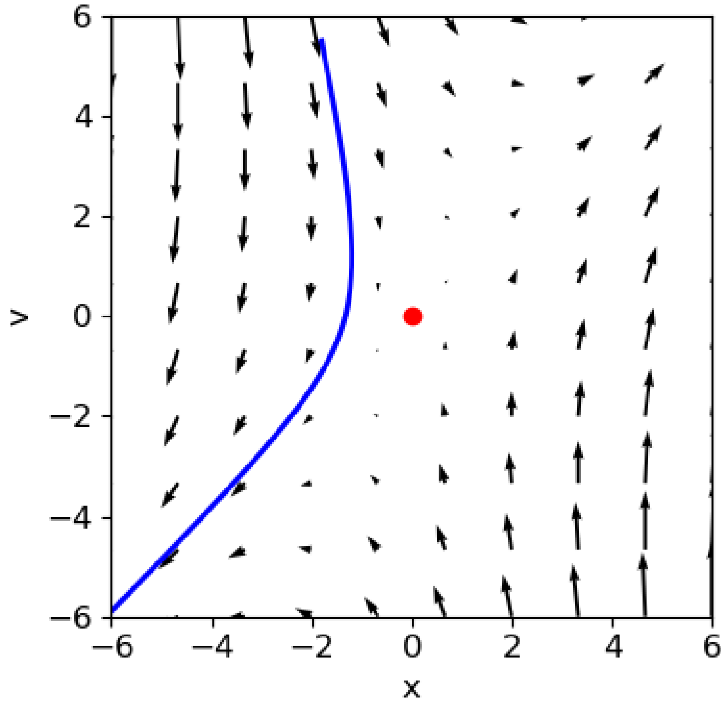

Fig. 3.3. Phase plane: quiver plot. The streamplot gives a much better view of the vector field.

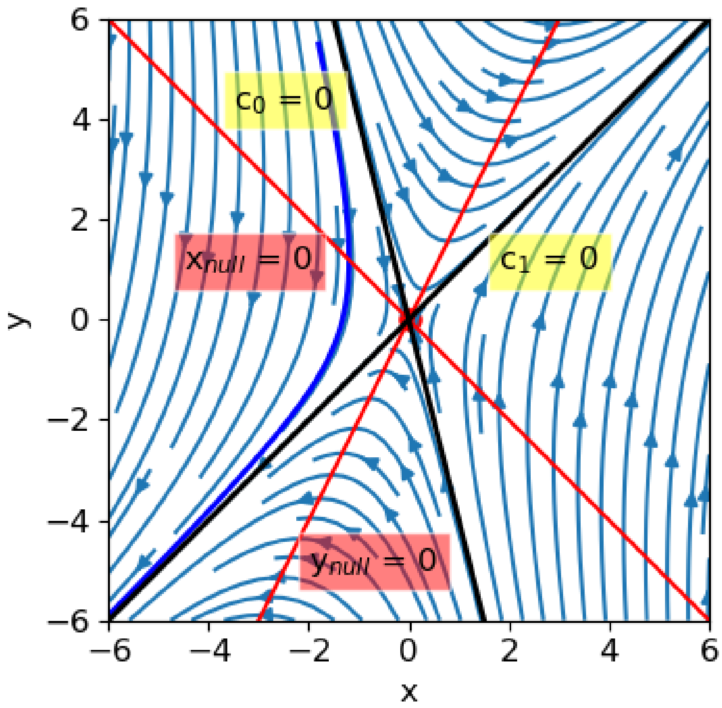

Fig. 3.4. Phase plane: streamline plot (vector field) and trajectory (blue). The x and y nullclines are shown in red. At the x nullcline the flow is only vertical and at the y nullcline the flow is horizontal. The c_0 and c_1 straight lines are shown in **black**. Along the c_1 straight line everything expands while along the c_0 straight line flow is pulled towards the saddle point (0, 0).