DOING PHYSICS WITH PYTHON

[2D] DYNAMICAL SYSTEMS

LINEAR PLANAR SYSTEMS

COMPLEX EIGENVALUES

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DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive

GitHub

cs200.py **ds1423.py**

Jason Bramburger

Linear Planar Systems - Dynamical Systems | Lecture 14

https://www.youtube.com/watch?v=b8eJb5uwNZI

Greek letters are often avoided and letters used in this paper are closely related to the letters used in the Python Code.

This paper considers [2D] linear dynamical systems which have complex eigenvalues.

INTRODUCTION

In a planar linear dynamical system, complex eigenvalues of the system's real matrix indicate oscillatory behaviour, with the real part of the eigenvalue determining the rate of growth or decay and the imaginary part dictating the frequency of oscillation. The eigenvalues will appear as complex conjugate pairs, and their corresponding eigenvectors will also be complex. Solutions will exhibit stable or unstable spirals around the Origin or the Origin acts a centre for the trajectories.

The real part of the eigenvalue *a*, determines the stability and nature of the critical point at the Origin.

$$\lambda = a \pm b\,j \quad b \neq 0$$

$$\exp(\lambda t) = \exp(a t + b\,t\,j) = \exp(a t)\exp(b t\,j)$$

$$\mathrm{Re}\left(\exp(b t\,j)\right) = \cos(b t) = \cos(\omega t) \quad b \equiv \omega$$

- $a > 1$ system grows unboundedly as the ttrajectories spiral outwards. **Unstable focus**
- $a < 1$ system decays as the trajectories spiral inwards, converging to the Origin. **Stable focus**
- $a = 1$ solutions maintain a constant magnitude.
- $a = 0$ trajectories are closed curves (circles or ellipses) and do not spiral into or out of the origin

$b \equiv \omega$ gives the angular frequency of the oscillation and means the system is spiralling rather than monotonically decaying to or growing from the Origin.

The **phase portrait** visually represents the system's behaviour, showing the trajectories of solutions in the *x*-*y* plane. With complex eigenvalues, the phase portrait shows solutions that move in spirals or ellipses around the Origin. The direction of the spiral (clockwise or counterclockwise) is determined by the sign of the imaginary part of the eigenvalue.

SIMULATIONS

$$\lambda = a + b\,j$$

Example 1 STABLE FOCUS *a* < 1

$\dot{x} = -1.0\,x - 1.0\,y \qquad \dot{y} = 1.0\,x - 1.0\,y$

A matrix: a11 = -1.0 a12 = -1.0 a11 = 1.0 a12 = -1.0

Determinant A = 2.00

Initial conditions (x0, y0)

(1.00, 4.50) (-2.90, 1.31) (-3.00, -4.00) (1.00, -2.00)

(3.00, -1.50) (1.50, -0.50) (0.10, 0.10)

Eigenvalues (-1+1j) (-1-1j)

Eigenfunctions [-0.+1.j 0.-1.j] [1.-0.j 1.+0.j]

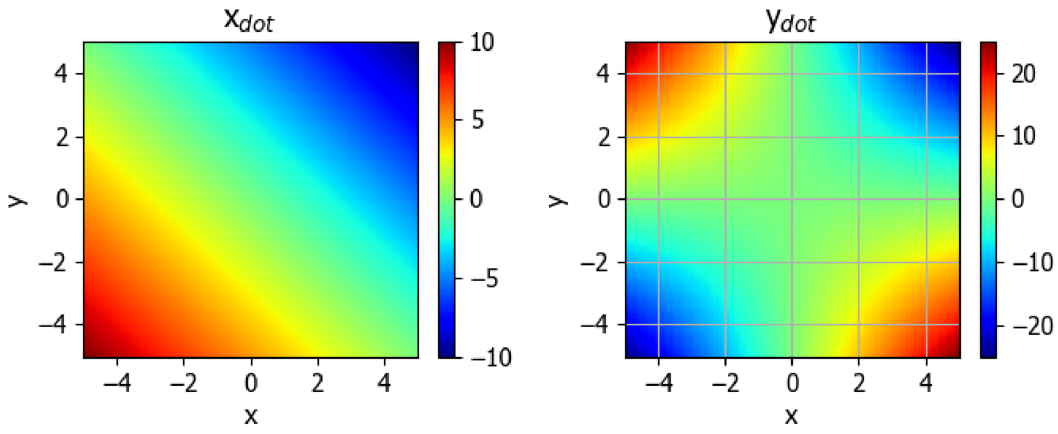

Fig. 1.1. [2D] view of the system equations.

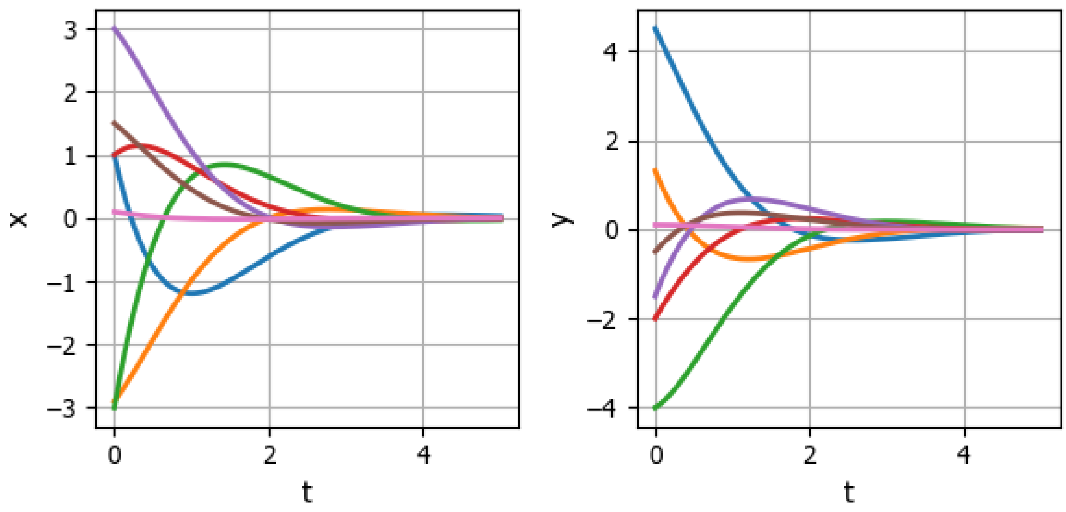

Fig. 1.2. Time evolution of the x and y trajectories. All trajectories are attracted to the stable fixed point at the Origin (0, 0).

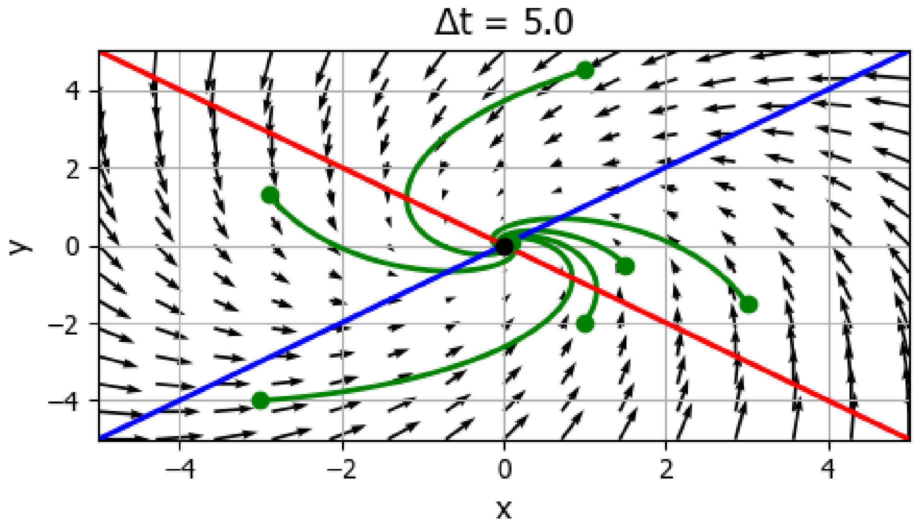

Fig. 1.3. Vector field spiral: quiver plot

Red: x-nullcline $(\dot{x} = 0)$ Blue: y-nullcline $(\dot{y} = 0)$

green: initial conditions (x_0, y_0) and trajectories

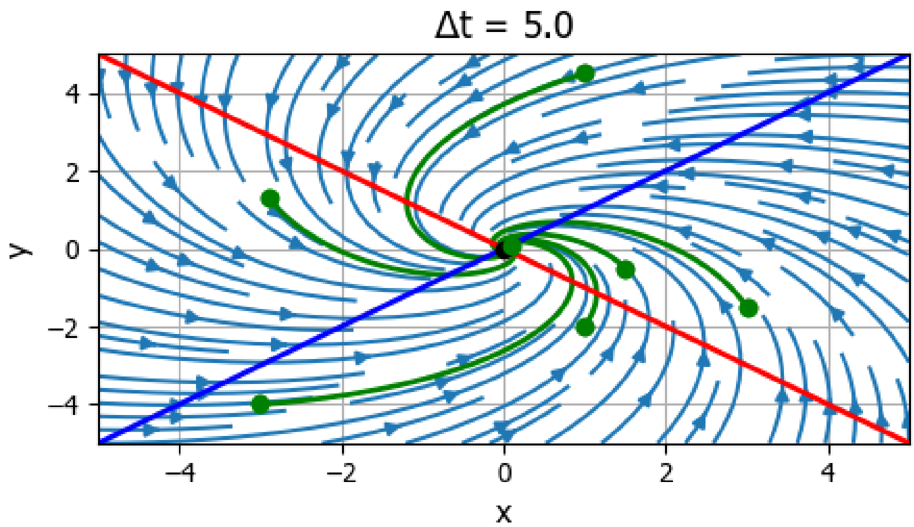

Fig. 1.4. Vector field spiral: streamplot.

All trajectories are attracted to the stable fixed point at the Origin (0, 0).

The eigenvalues and eigenvectors are complex solutions. The Origin is the only critical point and is a **stable focus**. The eigenvectors are complex and there are no real manifolds. If the real part of the eigenvalue is negative the trajectories will spiral into the Origin and in this case the equilibrium solution will be asymptotically stable..

Example 2 UNSTABLE FOCUS *a* > 1

$$\dot{x} = 0.03x + 0.09y \qquad \dot{y} = -0.04x - 0.03y$$

A matrix: a11 = 0.03 a12 = 0.09 a11 = -0.04 a12 = 0.03

Determinant A = 0.00450

Initial conditions (x0, y0)

(-0.0500, 0.0500)

(0.0500, -0.0500)

(0.0000, 0.0000)

(0.0000, 0.0000)

(0.0000, 0.0000)

Eigenvalues (0.03+0.06j) (0.03-0.06j)

Eigenfunctions [0.-1.5j 0.+1.5j] [1.+0.j 1.-0.j]

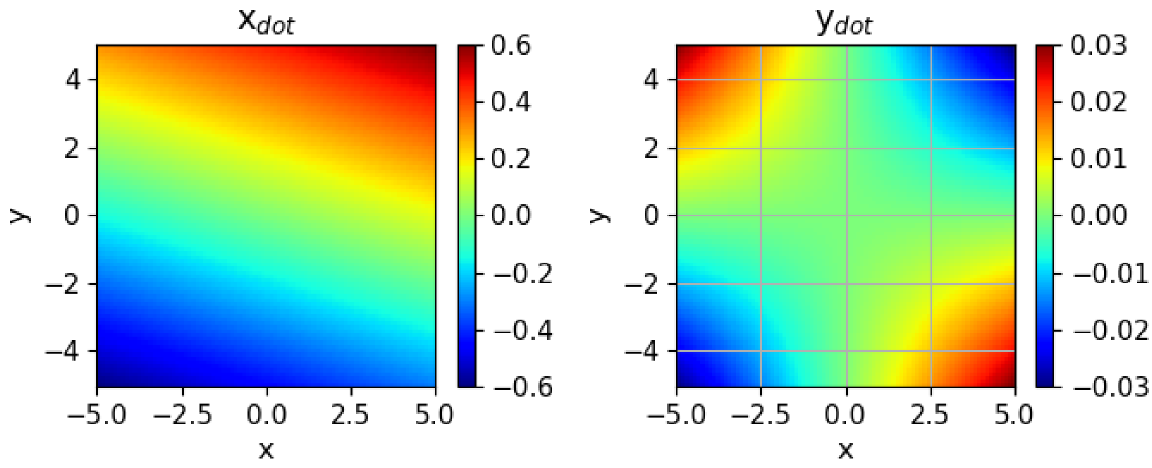

Fig. 2.1. [2D] view of the system equations.

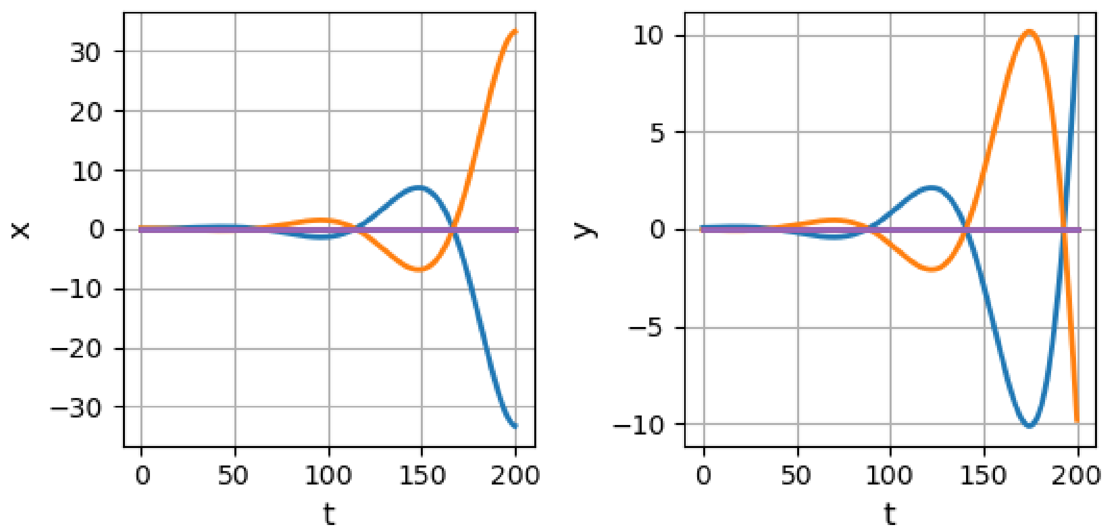

Fig. 2.2. Time evolution of the x and y parameters.

$$t \to \infty \quad |x(t)| \to \infty \quad |y(t)| \to \infty$$

Initially the flow oscillates before diverging to infinity.

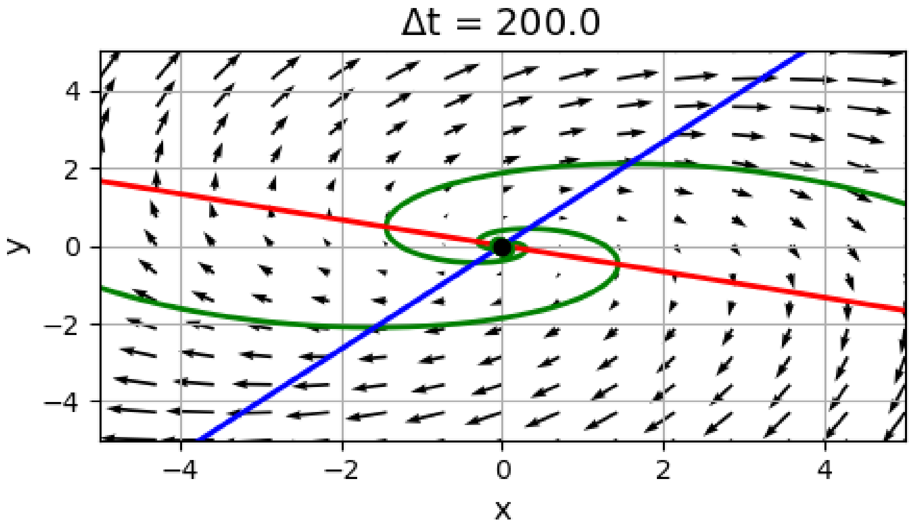

Fig. 2.3.   Vector field spiral: quiver plot.

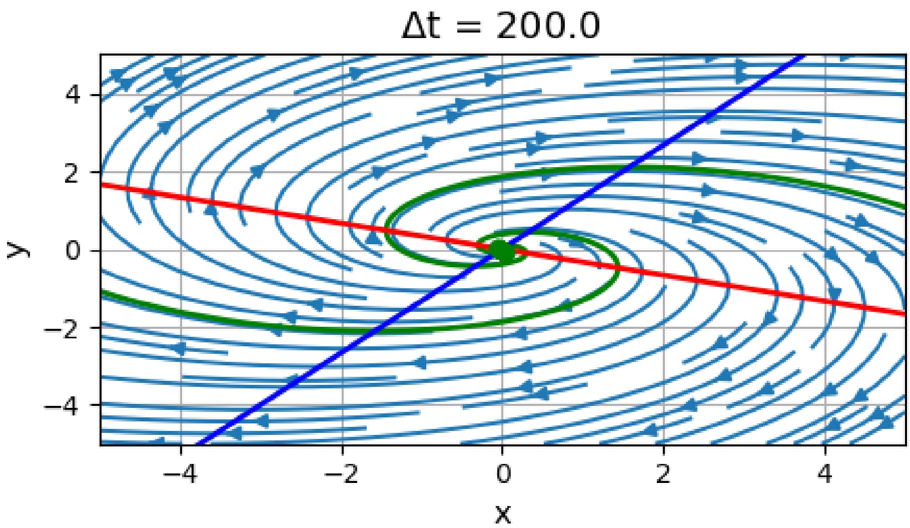

Fig. 2.4. Vector field spiral: streamplot.

If the real part of the eigenvalue is positive ($a > 0$) the trajectories will spiral away from the Origin and in this case the equilibrium solution will be asymptotically unstable. The fixed point at the Origin is an **unstable focus**.

$$t \to \infty \quad |x(t)| \to \infty \quad |y(t)| \to \infty$$

Example 3 UNSTABLE FOCUS *a* = 0

System: $\dot{x} = 3x + 9y \qquad \dot{y} = -4x - 3y$

A matrix: a11 = 3.00 a12 = 9.00 a11 = -4.00 a12 = -3.00

Determinant A = 27.00000

Initial conditions (x0, y0)

(1.0000, 1.0000)

(-2.0000, -2.0000)

Eigenvalues (0+5.196j) (0 -5.196j)

Eigenfunctions [-0.75-1.299j -0.75+1.299j] [1.+0.j 1.-0.j]

When the eigenvalues of a matrix of $\mathbf{J} \equiv \mathbf{A}$ are purely complex, as they are in this case, the trajectories of the solutions will be ellipses that are centred at the Origin. The only thing that we really need to concern ourselves with here are whether they are rotating in a clockwise or counterclockwise direction.

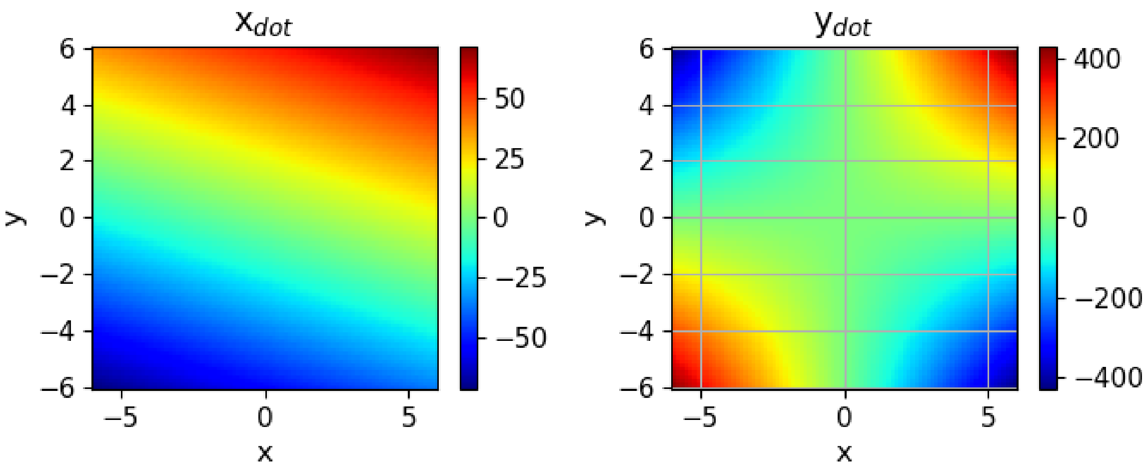

Fig. 3.1. [2D] view of the system equations.

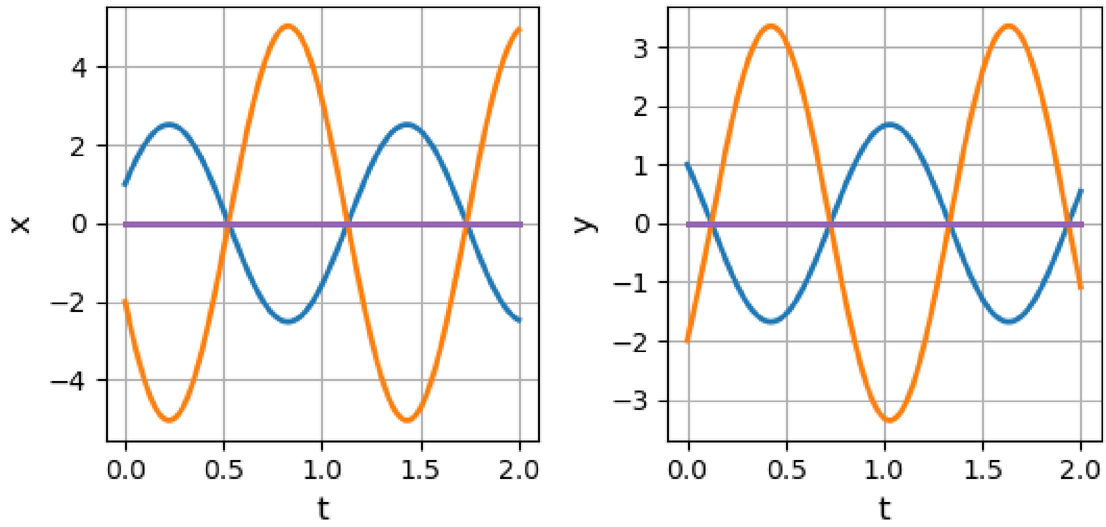

Fig. 3.2. Time evolution of the x and y parameters.

$$t \to \infty \quad |x(t)| \to \infty \quad |y(t)| \to \infty$$

From the plots, the period T of oscillation is 1.2. From the eigenvalues, the complex part b gives the frequency and hence the period. $b = \omega = 5.196 \quad T = \dfrac{2\pi}{\omega} = 1.2$

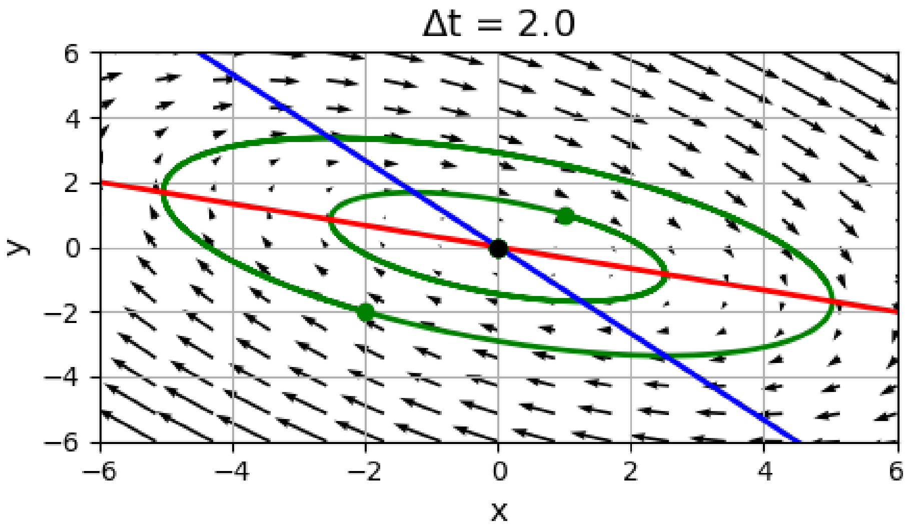

Fig. 3.3. Vector field: quiver plot. The red line is the x-nullcline ($\dot{x} = 0$), and the blue line is the y-nullcline ($\dot{y} = 0$). The direction of flow is clockwise. The flow is clockwise.

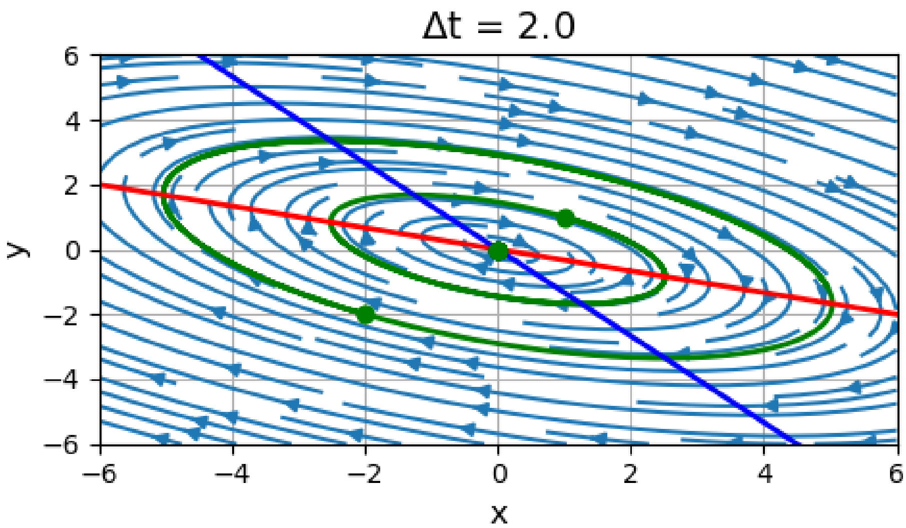

Fig. 3.4. Vector field: streamplot. The red line is the x-nullcline ($\dot{x} = 0$), and the blue line is the y-nullcline ($\dot{y} = 0$).

The Origin is the only critical point. In the phase portraits, the flow is horizontal on the line where $\dot{y} = 0$ and vertical on the line where $\dot{x} = 0$. The equilibrium solution in the case is called a **centre** and is stable.

The equilibrium solution is stable and not asymptotically stable. Asymptotically stable refers to the fact that the trajectories are moving in toward the equilibrium solution as t increases. The trajectories in this example are simply revolving around the equilibrium solution and not moving in towards it. The trajectories are also not moving away from the equilibrium solution and so they aren't unstable. Therefore, we call the equilibrium solution neutral stable.

References

https://tutorial.math.lamar.edu/Classes/DE/RepeatedEigenvalues.aspx