DOING PHYSICS WITH PYTHON

[2D] NON-LINEAR DYNAMICAL SYSTEMS
SADDLE-NODE BIFURCATIONS
REAL EIGENVALUES

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DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive

GitHub

cs120.py   cs121.py   cs122.py

INTRODUCTION

A saddle-node bifurcation in a dynamical system is a local bifurcation where two equilibrium points (fixed points) collide and then annihilate each other, or are created from nothing. This phenomenon is also known as a fold bifurcation, tangential bifurcation, or blue skies bifurcation (refers to the sudden creation of two fixed points from nothing). Two fixed points emerge or are destroyed when nullclines become tangent.

Collision and Annihilation: Two fixed points, one stable and one unstable, merge into a single semi-stable fixed point at the bifurcation point.

Creation of Fixed Points: As the bifurcation parameter changes, two new fixed points can appear at the bifurcation point.

SIMULATIONS

Consider the [2D] nonlinear dynamical system governed by the equations

$$\dot{x} = r - x^2 \qquad \dot{y} = -y \qquad f(x) = r - x^2 \quad g(y) = -y$$

where r is the bifurcation parameter. The fixed points of the system are dependent upon the bifurcation parameter r.

We need to consider the three cases when $r < 0$, $r = 0$ and $r > 0$ individually to explore the system dynamics for the x subsystem given the fact that in the y subsystem, the y-direction the motion is exponentially damped $(t \to \infty \Rightarrow y \to 0)$. Figure 1 shows the phase portrait plots for $r = 9 > 0$, $r = 0$ and $r = -9 < 0$. For $r > 0$, there are two fixed points $(\sqrt{r}, 0)$ which is a **node** (stable) and $(-\sqrt{r}, 0)$ which is a **saddle** (unstable). As r decreases, the saddle and stable node move closer and coalesce at $r = 0$ to give a semi-stable fixed point. When $r < 0$, the peak of the parabola falls below zero and all fixed points are annihilated.

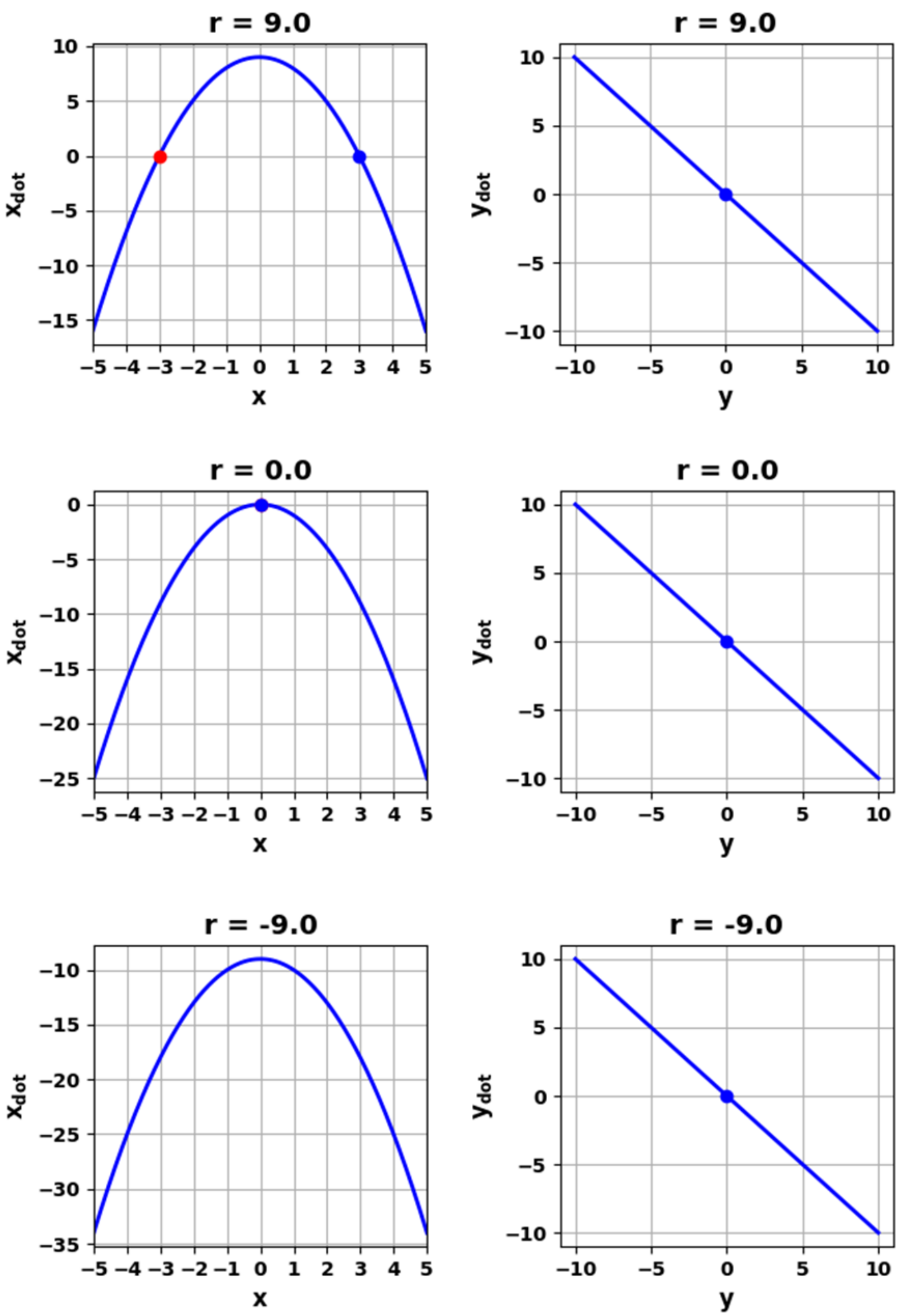

Fig. 1. As r decreases ($r > 0$) the stable and unstable fixed points move closer together and finally merge and annihilate each other at the bifurcation point $r = 0$ and for $r < 0$ no fixed points exist.

Mathematical and graphical analysis

System

$$\dot{x} = r - x^2 \qquad \dot{y} = -y \qquad f(x) = r - x^2 \quad g(y) = -y$$

Fixed points (x_e, y_e)

$$\dot{y} = 0 \quad y_e = 0 \qquad \dot{x} = r - x_e^{\ 2} = 0 \quad x_e = \pm\sqrt{r}$$

The system equation $\dot{y} = -y$ is independent of the control parameter r. For all initial values $y(0)$ will exponentially converge to $y = 0$

$$t \to \infty \quad y(t) \to 0$$

Stability

To determine the stabilities of a fixed point one needs to evaluate the Jacobian matrix of the system for local stability at (x_e, y_e) and find the eigenvalues. The Jacobian matrix is

$$\mathbf{J}(x,y) = \begin{pmatrix} \partial f / \partial x & \partial f / \partial y \\ \partial g / \partial x & \partial g / \partial y \end{pmatrix} = \begin{pmatrix} -2x & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathbf{J}(x_e,y_e) = \begin{pmatrix} \partial f / \partial x & \partial f / \partial y \\ \partial g / \partial x & \partial g / \partial y \end{pmatrix}_{x=x_e} = \begin{pmatrix} -2x_e & 0 \\ 0 & -1 \end{pmatrix}$$

We need to consider the three possibilities of the control parameter r: $r < 0$, $r = 0$, and $r > 0$.

Case 1: $\boldsymbol{r < 0}$ **cs123.py**

There are no fixed points since $\dot{x} < 0$ for all values of x

$$t \to \infty \quad x(t) \to -\infty \quad y(t) \to 0$$

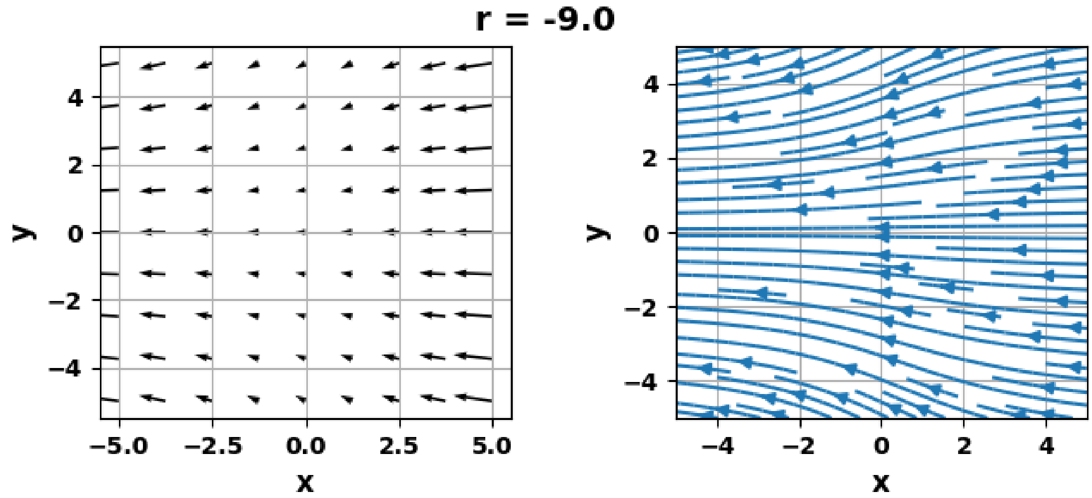

Fig. 1.1. Phase portrait (quiver and streamline plots)

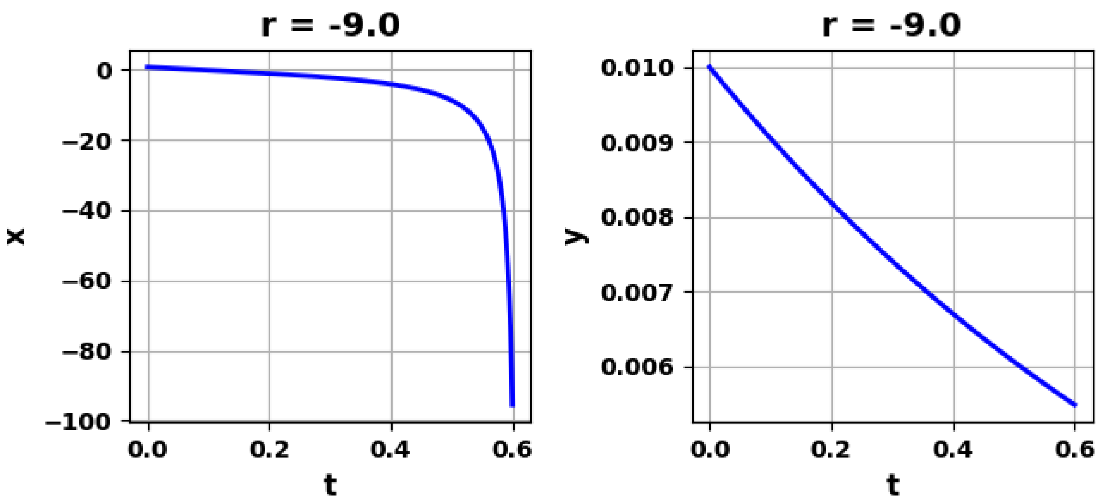

Fig 1.2. Trajectories; $t \to \infty \quad x(t) \to -\infty \quad y(t) \to 0$

Case 2: $\boldsymbol{r} = \mathbf{0}$ **cs122.py**

Fixed point (0, 0)

$$\dot{x} = -x^2 \quad \dot{y} = -y \qquad t \to \infty \quad x(t) \to -\infty \quad y(t) \to 0$$

$$\mathbf{J}_0 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

The eigenvalues of the Jacobian are (0, -1) This indicates that the fixed point (0, 0) is **semi-stable** and is a **saddle equilibrium**.

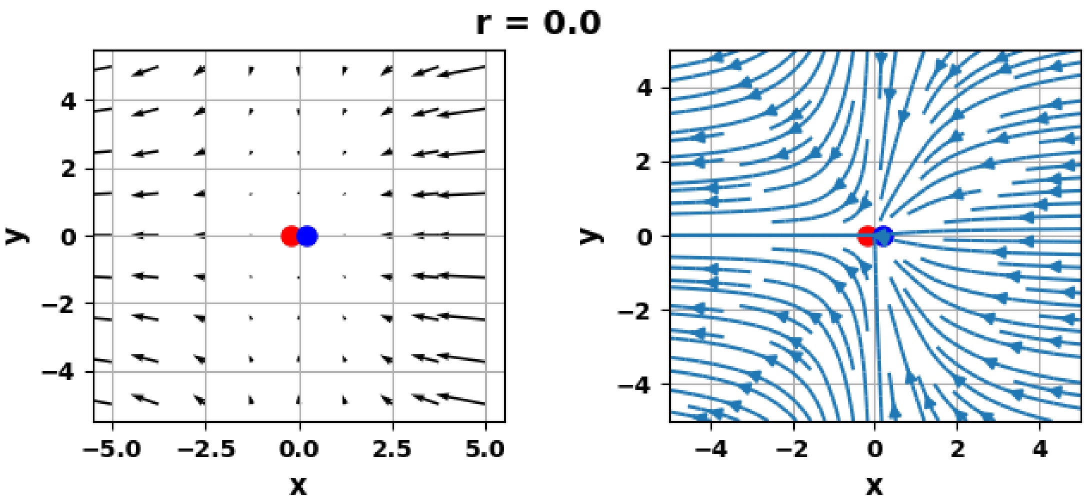

Fig 2.1. Phase portrait. The single fixed point is at the Origin (0, 0).

$$t \to \infty \quad x(0) < 0 \quad x(t) \to -\infty \quad y(t) \to 0$$
$$x(0) > 0 \quad x(t) \to 0 \quad y(t) \to 0$$

Using the phase portrait plots it is easy to predict the trajectory for any initial condition as the flow direction is given by an arrow in the quiver plot or by the tangent to a streamline.

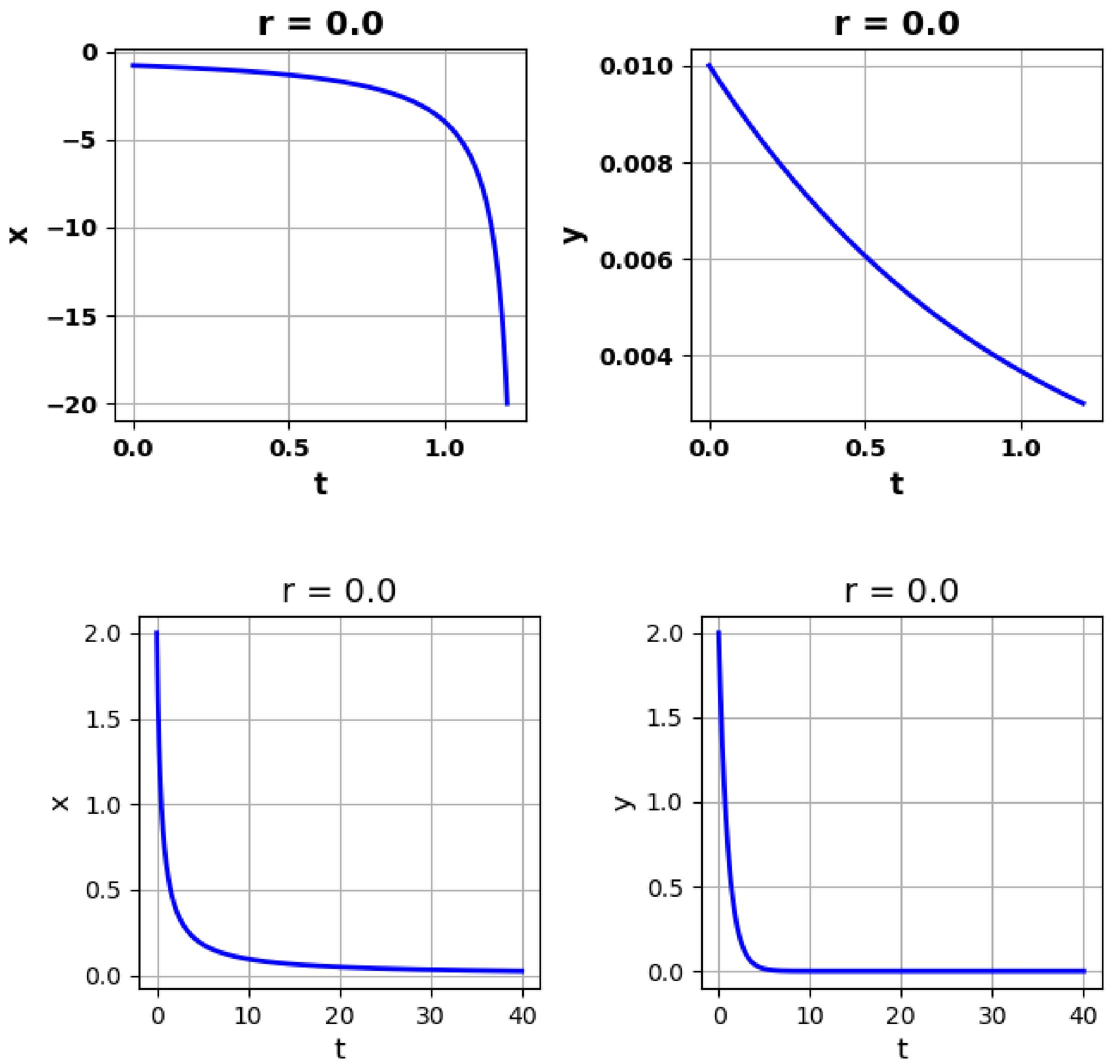

Fig 2.3. Trajectories. The single fixed point is at the Origin (0, 0).

$$t \to \infty \quad x(0) < 0 \quad x(t) \to -\infty \quad y(t) \to 0$$

$$x(0) > 0 \quad x(t) \to 0 \quad y(t) \to 0$$

Case 3: $r > 0$ **cs120.py**

The system has two fixed points: $\left(-\sqrt{r},\, 0\right)$ $\left(+\sqrt{r},\, 0\right)$

The Jacobian matrices are $\mathbf{J_P}$ and $\mathbf{J_M}$

$$x_e = +\sqrt{r} \quad \mathbf{J_P} = \begin{pmatrix} -2\sqrt{r} & 0 \\ 0 & -1 \end{pmatrix} \qquad x_e = -\sqrt{r} \quad \mathbf{J_M} = \begin{pmatrix} +2\sqrt{r} & 0 \\ 0 & -1 \end{pmatrix}$$

Consider the case when $r = 9$, then the fixed points and eigenvalues of the Jacobian are:

x_e = (-3, 0) eigenvalues = (+6, -1)

The eigenvalues are real (positive, negative) therefore the fixed point is a **saddle**. and is **unstable**.

x_e = (+3, 0) eigenvalues = (-6, -1)

The eigenvalues are real (negative, negative) therefore the fixed point is a **stable node**.

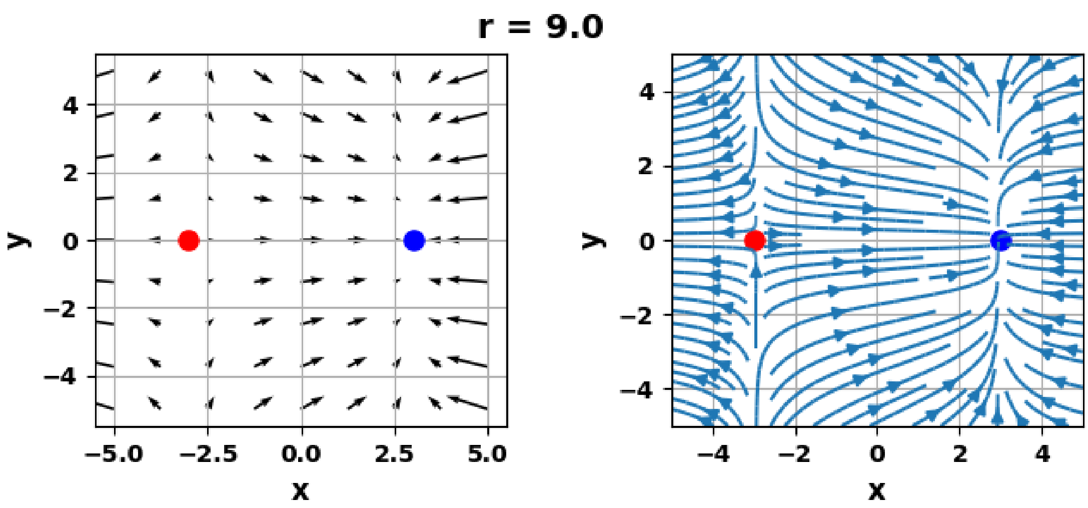

Fig . 3.1. Phase portraits: Vector fields.

The fixed point (-3, 0) is unstable and the fixed point (+3,0) is stable.

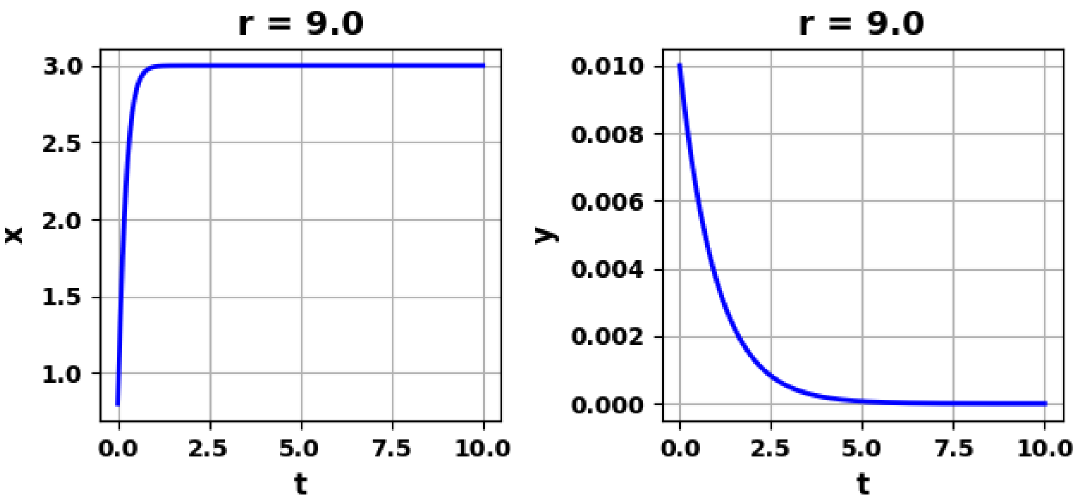

Fig. 3.2. Trajectories

$$x(0) = 0.50 \;\; y(0) = 0.01 \quad t \to \infty \quad x(t) \to 3.0 \;\; y(t) \to 0$$

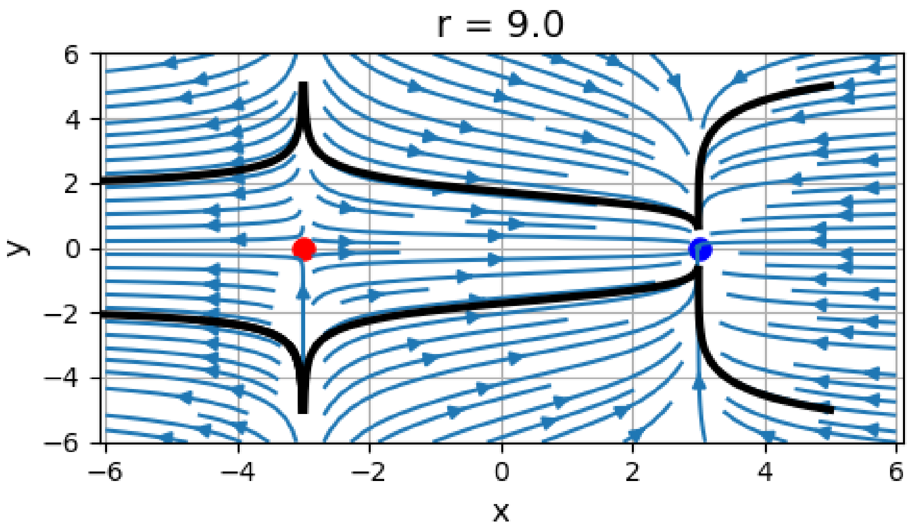

Fig. 3.3. Phase portrait and trajectories

$$\left(x(0) = 5.00 \;\; y(0) = 5.00 \quad t_2 = 2\right)$$
$$\left(x(0) = 5.00 \;\; y(0) = -5.00 \quad t_2 = 2\right)$$
$$\left(x(0) = -3.01 \;\; y(0) = 5.00 \quad t_2 = 1\right)$$
$$\left(x(0) = 5.00 \;\; y(0) = 5.00 \quad t_2 = 2\right)$$

$$\left(x(0)=-3.01 \;\; y(0)=-5.00 \;\; t_2=1\right)$$
$$\left(x(0)=-3.01 \;\; y(0)=-5.00 \;\; t_2=1\right)$$
$$\left(x(0)=-2.99 \;\; y(0)=5.00 \;\; t_2=1\right)$$
$$\left(x(0)=-2.99 \;\; y(0)=-5.00 \;\; t_2=1\right)$$

The fixed point (-3,0) is unstable. For two initial conditions surrounding this unstable fixed point, the trajectories maybe very different.

$$x(0)=-3.01 \;\; y(0)=5 \quad t\to\infty \;\; x(t)\to-\infty \;\; y(t)\to 0$$

$$x(0)=-2.99 \;\; y(0)=5 \quad t\to\infty \;\; x(t)\to+3 \;\; y(t)\to 0$$

From the graphical analysis, we see that the system with $r > 0$ has two fixed points, one is a stable node $\left(\sqrt{r},0\right)$ and the other is a saddle point. When r decreases, the saddle and the stable node approach each other. They collide at $r = 0$ and disappear when $r < 0$. This type of bifurcation is known as **saddle-node bifurcation** (figure 3.4). The name “saddle-node” is because its basic mechanism is the collision of two fixed points - a saddle and a node of the system and in this example the bifurcation point is $r = 0$.

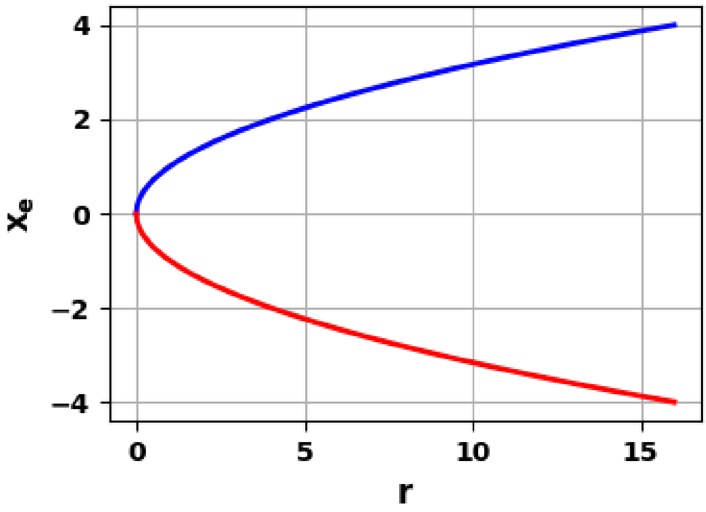

Fig. 3.4. Bifurcation diagram

Even after the fixed points have annihilated each other, they continue to influence the flow as they leave a ghost, a bottleneck region that sucks trajectories in and delays them before allowing passage out the other side. Bifurcation theory is rife with conflicting terminology, and different people use different words for the same thing. For example, the saddle-node bifurcation is sometimes called a fold bifurcation.

[2D] VIEW OF SYSTEM EQUATIONS

$$\dot{x} = 9 - x^2 \qquad \dot{y} = -y$$

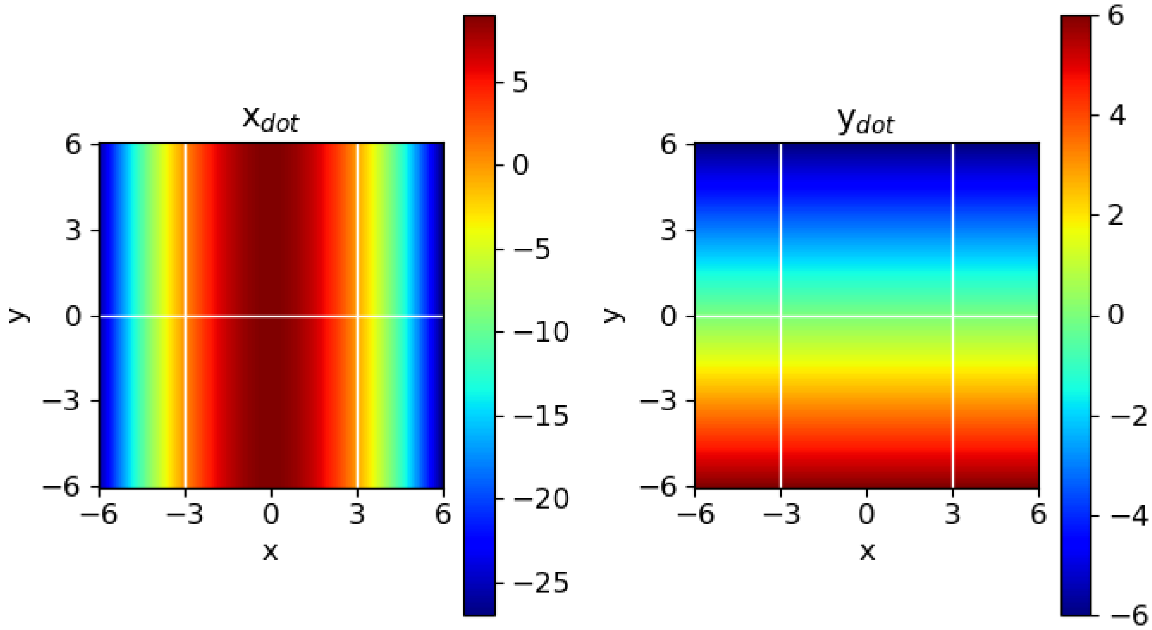

Fig 4.1. [2D] view of system equations $\dot{x}$ $\dot{y}$

$\dot{x}$ (x_{dot}) > 0 flow is in the + x direction →

$\dot{x}$ (x_{dot}) < 0 flow is in the - x direction ←

$\dot{x}$ (x_{dot}) $= 0$ ← x_e = -3 → (unstable)

→ $x_e = +3$ (stable) ←

$\dot{y}$ (y_{dot}) > 0 flow is in the + y direction ↑

$\dot{y}$ (y_{dot}) < 0 flow is in the - y direction ↓

↑

$\dot{y}$ (y_{dot}) $= 0$ $y_e = 0$ (stable)

↓

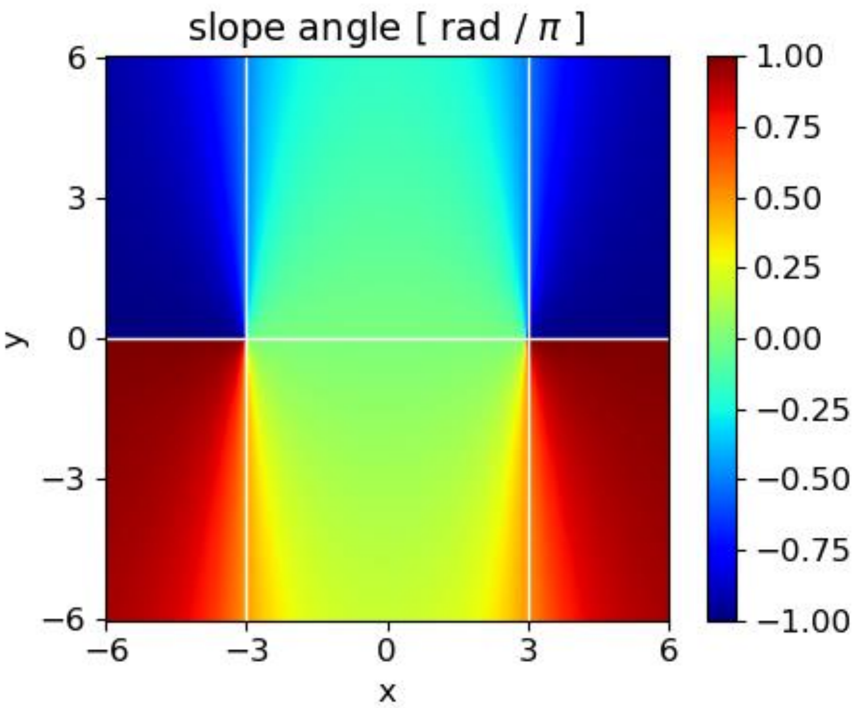

Fig. 4.2. Slope angle θ.

$\theta = 0 \rightarrow$ $\theta = 0.5 \uparrow$ $\theta = -0.5 \downarrow$ $\theta = -1 \leftarrow$ $\theta = +1 \leftarrow$

The slope function and its slope angle are $dy(x, y)/dx = \tan\theta$ where θ is expressed in rad/π. Therefore $-1 \le \theta \le +1$.

The flow is towards the stable fixed point at (+3,0) and way from the unstable fixed point (-3, 0).