DOING PHYSICS WITH PYTHON

[2D] NON-LINEAR DYNAMICAL SYSTEMS

TRANSCRITICAL BIFURCATIONS

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cs123.py cs124.py cs125.py

TRANSCRITICAL BIFURCATIONS

There are certain scientific situations where a fixed point must exist for all values of a parameter and can never be destroyed. For example, in the logistic equation and other simple models for the growth of a single species, there is a fixed point at zero population, regardless of the value of the growth rate. However, such a fixed point may change its stability as the parameter is varied. The **transcritical bifurcation** is the standard mechanism for such changes in stability.

The normal form for a transcritical bifurcation is

$$\dot{x} = r\,x - x^2 \qquad \dot{y} = -y$$

We need to consider the three cases when $r < 0$, $r = 0$ and $r > 0$ individually to explore the system dynamics for the x subsystem given the fact that in the y subsystem, the y-direction the motion is exponentially damped $\left(t \to \infty \Rightarrow y \to 0\right)$.

When $r \neq 0$ there are two fixed points $(r, 0)$ and $(0, 0)$.
When $r = 0$ there is only one fixed point $(0, 0)$.
Hence, there is always a fixed point at the Origin $(0, 0)$.

For $r < 0$, there is an unstable fixed point at $x_e = \mathrm{r}$ and a stable fixed point at $x_e = 0$. As r increases, the unstable fixed point approaches the Origin, and coalesces with it when $r = 0$. Finally, when $r > 0$, the Origin has become unstable $x_e = 0$, and $x_e = r$ is now stable. Hence, we can say that an exchange of stabilities has taken place between the two fixed points. Note the important difference between the saddle-node and transcritical bifurcations. In the transcritical case, the two fixed points don't disappear after the bifurcation—instead they just switch their stability.

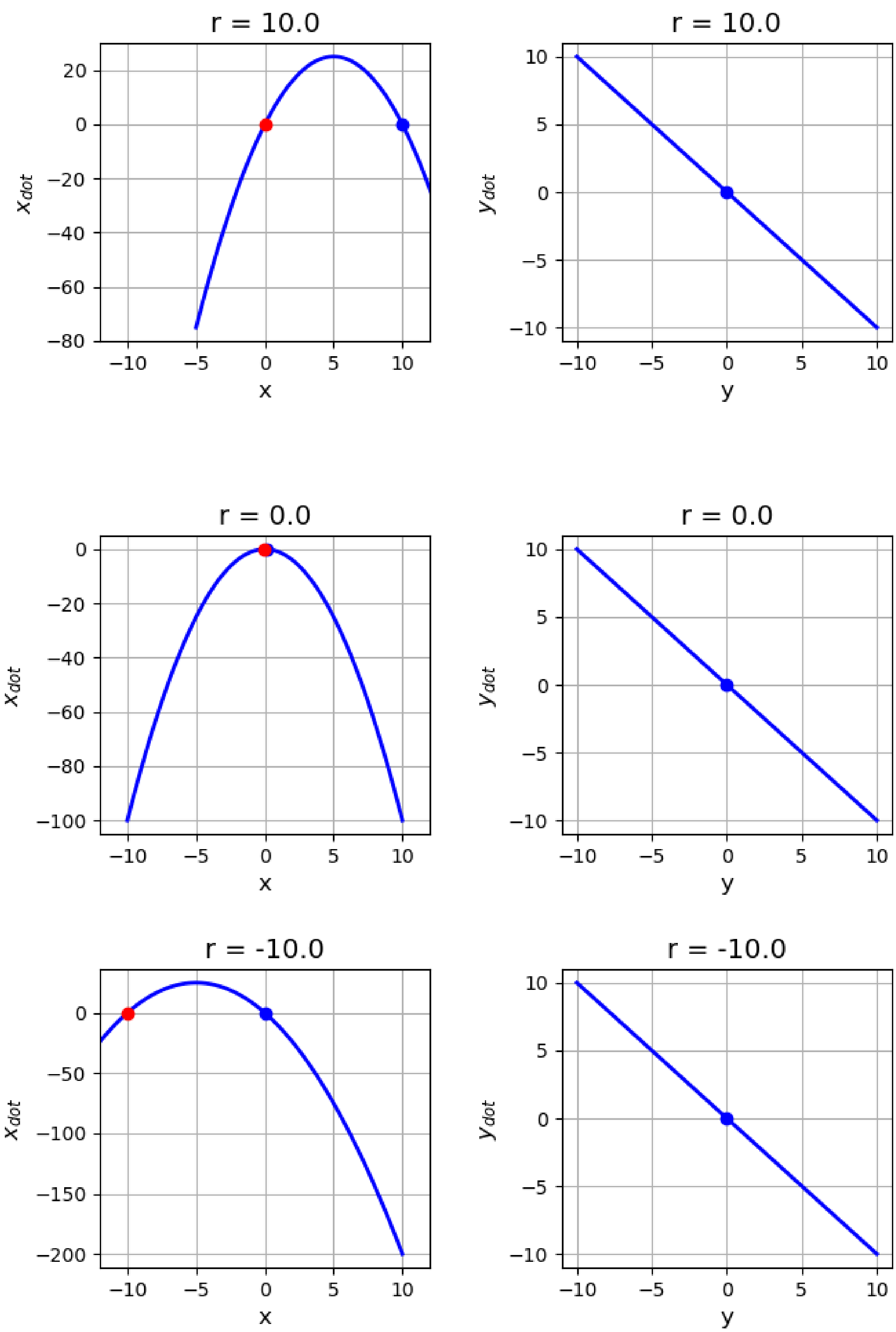

Fig. 1. The stability of the fixed point at the Origin (0, 0) depends on the sign of the *r* parameter.

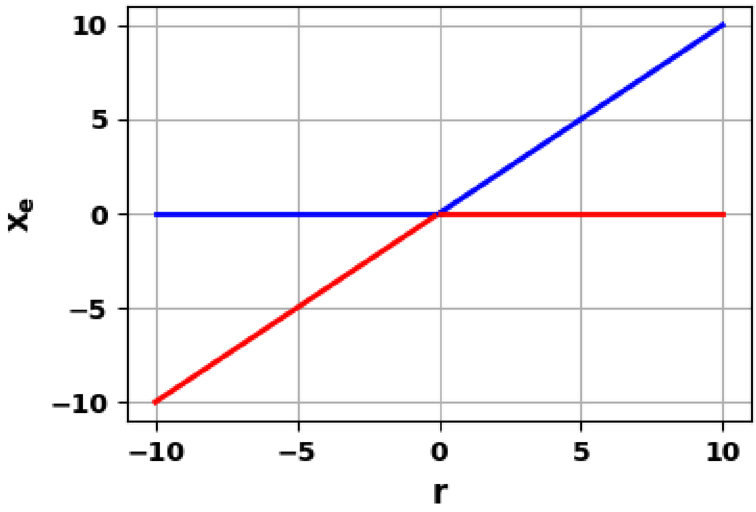

Fig. 2. Bifurcation diagram for x subsystem.

Mathematical analysis

Fixed points: $\dot{x} = r\,x_e - x_e^{\,2} = 0 \quad x_e = 0 \quad x_e = r$

Stability: To determine the stabilities of the fixed points, one needs to evaluate the Jacobian matrix of the system for local stability and find the eigenvalues. The Jacobian matrix is

$$\mathbf{J}(x_e, y_e) = \begin{pmatrix} r - 2x & 0 \\ 0 & -1 \end{pmatrix}$$

$r > 0$ **cs125.py**

The system has two fixed points: $(0,\,0)$ $(r,\,0)$

The Jacobian matrices are

$$x_e = 0 \quad \mathbf{J} = \begin{pmatrix} r & 0 \\ 0 & -1 \end{pmatrix} \qquad x_e = r \quad \mathbf{J} = \begin{pmatrix} -r & 0 \\ 0 & -1 \end{pmatrix}$$

Let $r = +10$, then the two fixed points are (0, 0) and (10,0) and the Jacobians matrices are

$$\mathbf{J}(0,0) = \begin{pmatrix} 10 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{eigenvalues} = (10, -1)$$

$$\mathbf{J}(10,0) = \begin{pmatrix} -10 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{eigenvalues} = (-10, -1)$$

- The fixed point (0, 0) is an unstable saddle point for $r > 0$.
- The fixed point (r, 0) is stable node for $r > 0$.

$r = 0$ **cs124.py**

Let $r = 0$, then there is one fixed points (0, 0)

Jacobian is

$$\mathbf{J}(x,y) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

and the eigenvalues are (0, -1). The fixed point (0, 0) is **semi-stable**.

$r < 0$ cs123.py

Let r = -10, then the two fixed points are (0, 0) and (-10,0) and the Jacobians matrices are

$$\mathbf{J}(0,0) = \begin{pmatrix} -10 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{eigenvalues} = (-10, -1)$$

$$\mathbf{J}(-10,0) = \begin{pmatrix} +10 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{eigenvalues} = (+10, -1)$$

- The fixed point (0, 0) is a stable node for $r < 0$.
- The fixed point (r, 0) is a saddle point (unstable) for $r < 0$.

Graphical analysis

Figure 2.3 shows the **vector field** of the system as a Python quiver plot and as a streamplot. Figure 2.4 shows the time evolution of the system for different initial conditions. The ODEs were solved using the Python function **odeint**.

From the plots in figures 2.3 and 2.4, we see that the behaviour of the system changes when the bifurcation parameter r increases from a negative value and passes through the Origin ($r = 0$), where the saddle becomes a stable node and the stable node becomes a saddle.

r	(0,0)	(r, 0)
$r < 0$	stable node	unstable saddle
$r = 0$	semi-stable	
$r > 0$	unstable saddle	stable node

This type of bifurcation is known as **transcritical bifurcation**. and the bifurcation point is $r = 0$. This type of bifurcation is same as in a [1D] system where no fixed points are disappeared.

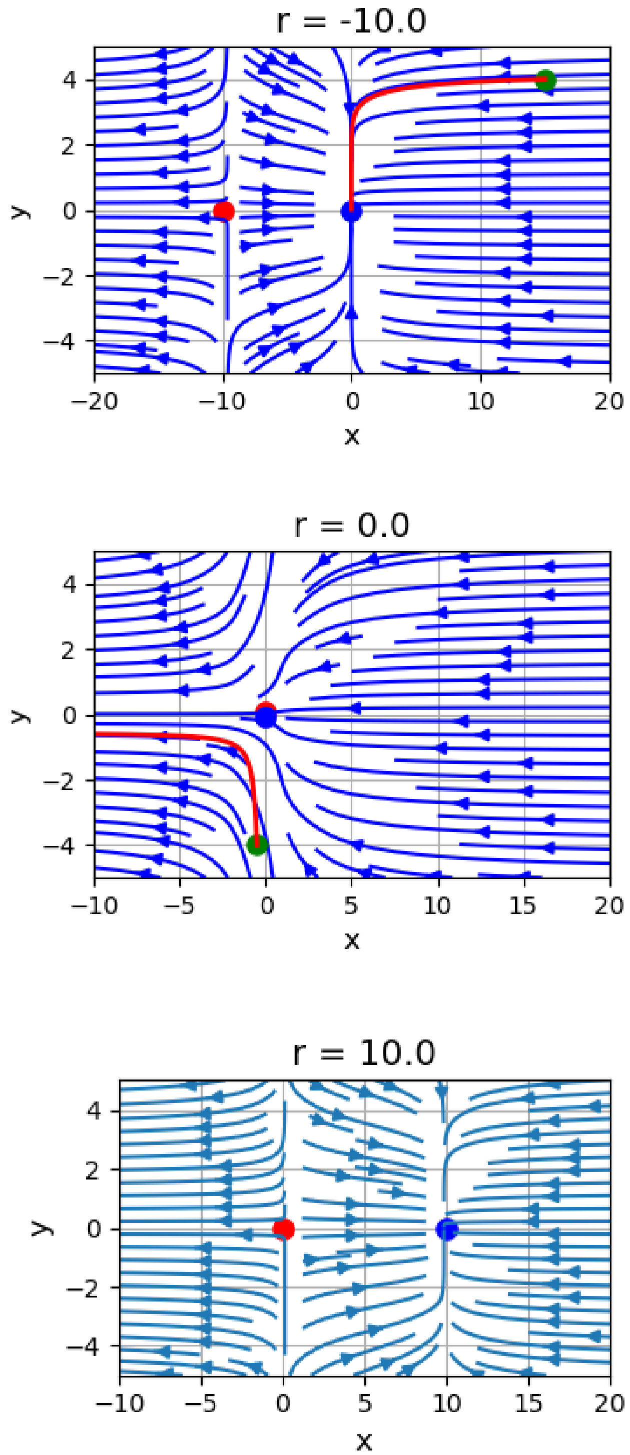

Fig. 3. phase portraits.

https://courses.physics.ucsd.edu/2009/Spring/physics221a/LECTURES/CH02_BIFURCATIONS.pdf

https://courses.physics.ucsd.edu/2020/Fall/physics200a/lectures.html