DOING PHYSICS WITH PYTHON

[2D] NON-LINEAR DYNAMICAL SYSTEMS
PITCHFORK BIFURCATIONS

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cs126.py cs127.py cs128.py

PITCHFORK BIFURCATIONS

The pitchfork bifurcation is common in physical problems that have a symmetry. For example, many problems have a spatial symmetry between left and right. In such cases, fixed points tend to appear and disappear in symmetrical pairs. For example, consider a vertical beam loaded at the top. For a small load, the beam is stable corresponding to zero horizontal deflection. But, if the load exceeds the buckling threshold, the beam may buckle to either the left or the right. The vertical position has gone unstable, and two new symmetrical fixed points, corresponding to left- and right-buckled configurations, have been born. There are two very different types of pitchfork bifurcation. The simpler type is called supercritical, and will be discussed first.

Example 1 Supercritical pitchfork bifurcation

Codes: **cs126.py** ($r < 0$), **cs127.py** ($r = 0$) and **cs128.py** ($r > 0$)

Consider a [2D] parametric system given by

$$\dot{x} = r\,x - x^3 \quad \dot{y} = -y$$

Note that this equation for x is invariant under the change of variables $x \rightarrow -x$. That is, if we replace x by $-x$ and then cancel the resulting minus signs on both sides of the equation, the equation does not change. This invariance is the mathematical expression of the left-right symmetry mentioned earlier and the vector fields are equivalent.

Mathematical analysis and graphical analysis

Fixed points: $\dot{x} = 0 \Rightarrow x_e = 0 \;\; x_e = \pm\sqrt{r} \quad \dot{y} = 0 \Rightarrow y_e = 0$

There is one fixed points when $r < 0$: (0,0)

There is one fixed points when $r = 0$: (0,0)

There are three fixed points when $r > 0$

$$(0,\,0),\ (+\sqrt{r}\,,\,0),\ \text{and}\ (-\sqrt{r}\,,\,0)$$

The Jacobian of the system is

$$\mathbf{J}(x,y)=\begin{pmatrix}\partial f/\partial x & \partial f/\partial y\\ \partial g/\partial x & \partial g/\partial y\end{pmatrix}=\begin{pmatrix}r-3x^2 & 0\\ 0 & -1\end{pmatrix}$$

***r* < 0** There is only one stable fixed point (0, 0)

Let r = -9 $\mathbf{J}(0,0)=\begin{pmatrix}-9 & 0\\ 0 & -1\end{pmatrix}$ eigenvalues $=(-9,-1)$

Both eigenvalues are negative, therefore the fixed point is **stable**.

***r* = 0** There is only one stable fixed point (0, 0)

Let $r=0$ $\mathbf{J}(0,0)=\begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}$ eigenvalues $=(0,-1)$

A dynamical system with one zero eigenvalue and one negative eigenvalue is stable but not asymptotically stable, meaning it stays near the equilibrium point under small perturbations but doesn't necessarily return to it over time, and is generally unstable in the strict sense as the system will not return to the Origin if the perturbation is along the direction of the zero eigenvalue. The zero eigenvalue indicates a degenerate case where there are infinite equilibrium points (there isn't a unique equilibrium point). A perturbation along the eigenvector associated with the zero eigenvalue will leave the system in a new equilibrium state.

The negative eigenvalue suggests a stable direction. If perturbed, the system will decay towards zero along the eigenvector corresponding to this negative eigenvalue. The negative eigenvalue ensures that the system will remain within a bounded region around the equilibrium if perturbed.

The zero eigenvalue introduces a lack of a unique equilibrium and the possibility of remaining in a different equilibrium state, while the negative eigenvalue provides a stable decay towards the Origin within that system.

r > 0 There are three fixed point:

(0, 0) is unstable

$\left(-\sqrt{r},0\right)$, and $\left(+\sqrt{r},0\right)$ are both stable

Let $r=9$ $\mathbf{J}(\pm 3,0)=\begin{pmatrix} -6 & 0 \\ 0 & -1 \end{pmatrix}$ eigenvalues $=\left(-6,-1\right)$ **stable**

$\mathbf{J}(0,0)=\begin{pmatrix} 9 & 0 \\ 0 & -1 \end{pmatrix}$ eigenvalues $=\left(9,-1\right)$ **unstable**

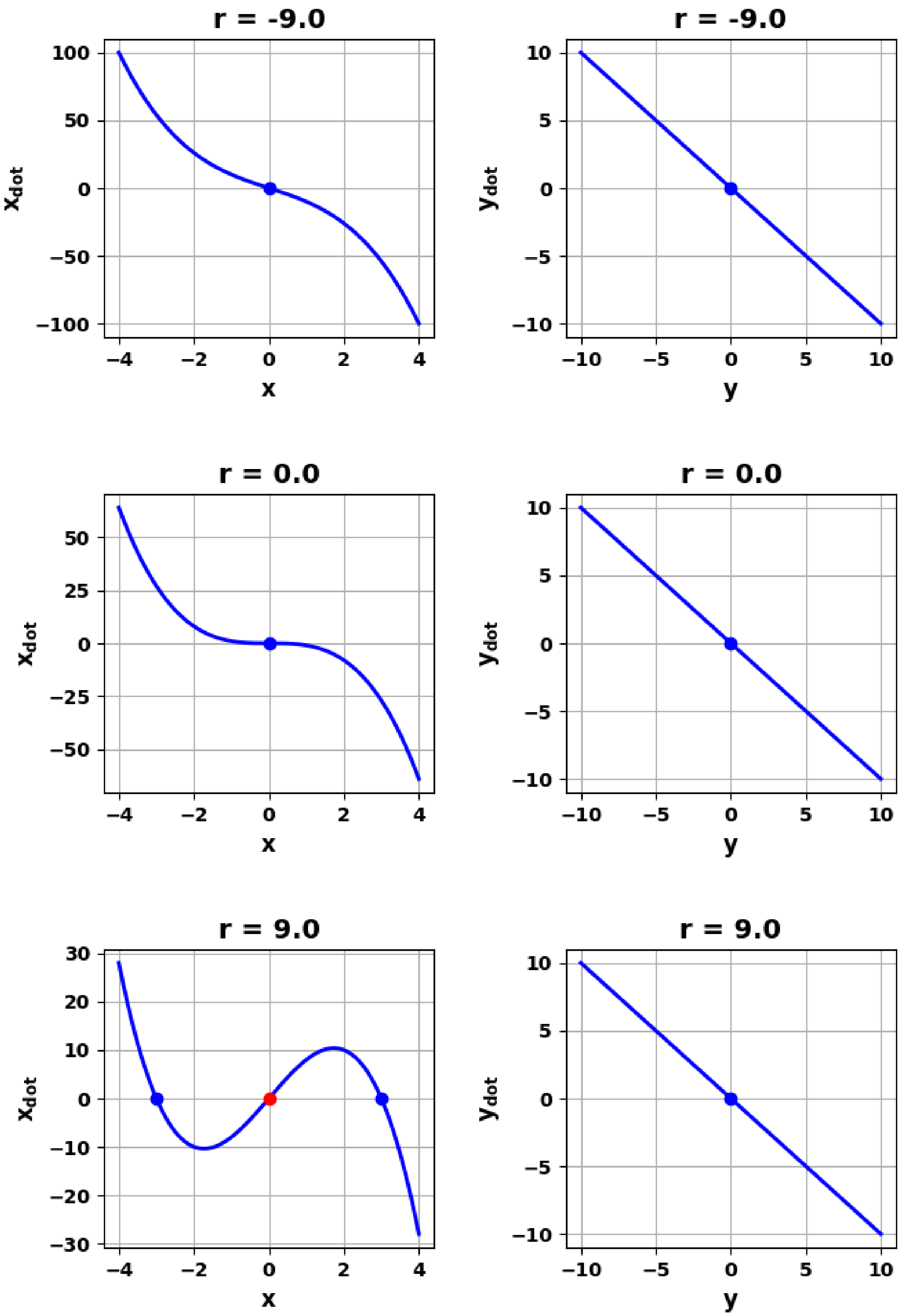

Fig. 1 The fixed points of the system for $r < 0$, $r = 0$ and $r > 0$.

Red dot unstable, blue dots are stable

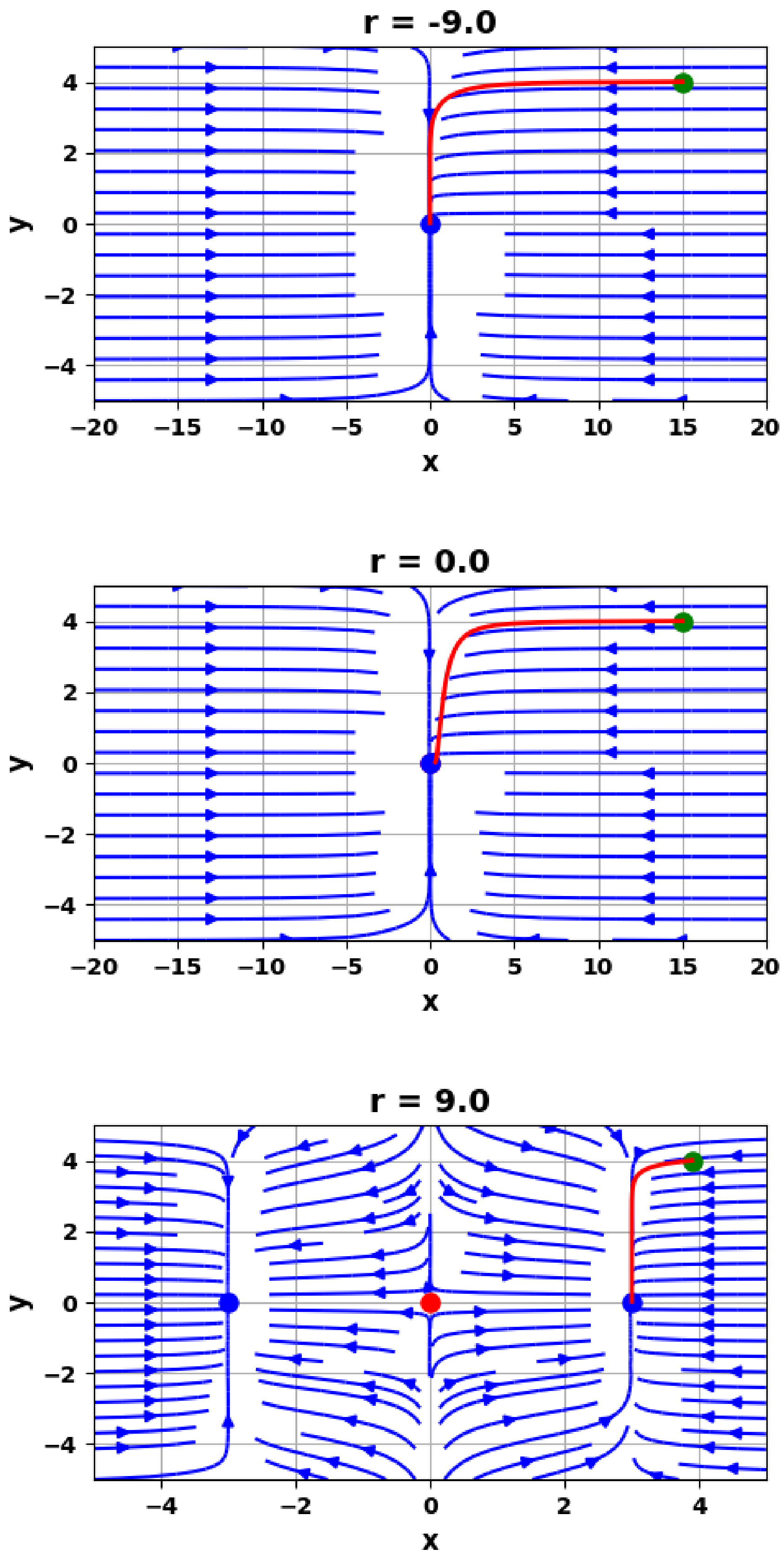

Fig. 2 Phase portraits.

When $r < 0$, the Origin is the only fixed point, and it is stable. When $r = 0$, the Origin is still stable, but much weaker. Now solutions no longer decay exponentially fast—instead the decay is a much slower. This lethargic decay is called critical slowing down. Finally, when $r > 0$, the Origin has become unstable. Two new stable fixed points appear on either side of the origin, symmetrically located at $x_e = \pm\sqrt{r}$. The reason for the term "pitchfork" becomes clear when we plot the bifurcation diagram (figure 3). Actually, pitchfork trifurcation might be a better word! This type of bifurcation is known as **supercritical pitchfork bifurcation**.

- $r < 0$, the only fixed point is a stable node at the Origin.
- $r = 0$, the Origin is still stable, but now we have very slow (algebraic) decay along the X-direction instead of exponential decay; this is the phenomenon of "critical slowing down.
- $r > 0$, the Origin loses stability and gives birth to two new stable fixed points symmetrically located at $\left(\pm\sqrt{r}, 0\right)$.

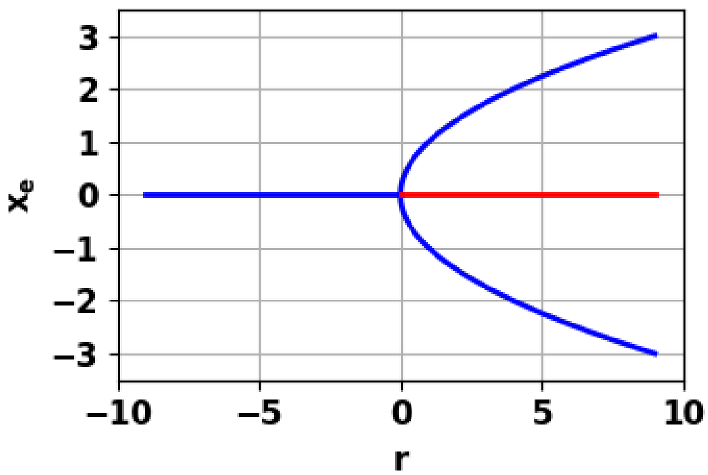

Fig. 3. Supercritical bifurcation diagram. Bifurcation point r = 0.

Example 2 Subcritical pitchfork bifurcation

In the supercritical case $\dot{x} = r\,x - x^3 \quad \dot{y} = -y$ (Example 3A), the cubic term is stabilizing and it acts as a restoring force that pulls $x(t)$ back toward $x = 0$. If instead the cubic term were destabilizing, as in

$$\dot{x} = r\,x + x^3 \quad \dot{y} = -y$$

then we'd have a **subcritical pitchfork bifurcation**.

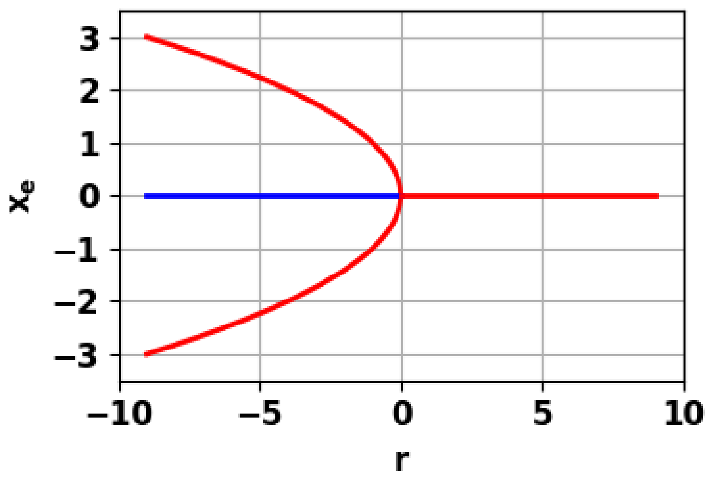

Fig. 4. Subcritical pitchfork bifurcation. Bifurcation point r = 0

https://courses.physics.ucsd.edu/2009/Spring/physics221a/LECTURES/CH02_BIFURCATIONS.pdf

https://courses.physics.ucsd.edu/2020/Fall/physics200a/lectures.html