DOING PHYSICS WITH PYTHON

[2D] NON-LINEAR DYNAMICAL SYSTEMS

SUPERCRITICAL HOPF BIFURCATIONS

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ds1509.py

A **Hopf bifurcation** is said to occur when varying a parameter of the system causes the set of solutions (trajectories) to change from being attracted to or repelled by a fixed point to a stable orbit.

Consider a two-dimensional system expressed in polar coordinates (R, θ) with control (bifurcation) parameter r

$$\dot{R} = r\,R - R^3 \quad \dot{\theta} = \omega$$

where R $(R \geq 0)$ is the radius vector of the trajectory, θ is the azimuthal angle, and ω is the angular frequency. The Cartesian coordinates (x, y) and period T of oscillation are

$$x = R\cos\theta \quad y = R\sin\theta \quad R^2 = x^2 + y^2 \quad \tan\theta = y / x$$

$$T = 2\pi / \omega$$

This is a very simple example of a supercritical Hopf bifurcation.

A good starting point to investigate the dynamics of the system is to plot the system equation $\left(R, \dot{R}\right)$ as shown in figure 1. In the analysis of the system dynamics, you need to consider the three cases $r < 0$, $r = 0$ and $r > 0$.

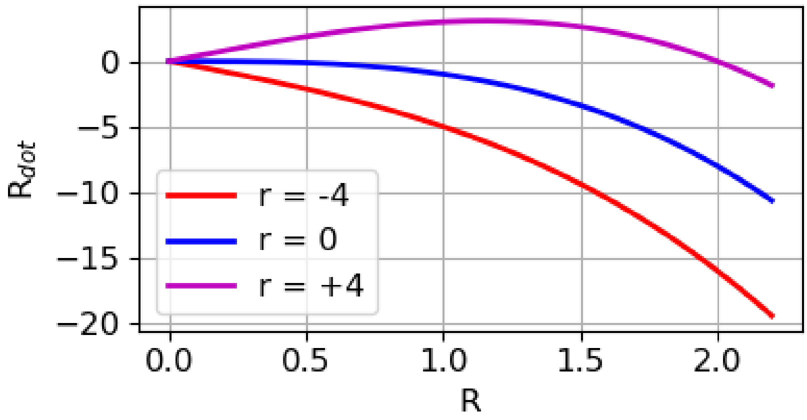

Fig. 1. Plot of the R subsystem equation. Stable values of the radius R_{ss} occur when $\dot{R} = 0$. The Origin (0, 0) is a fixed point for all values of r. When $r = 4$, the two stable radii are $R_{ss} = 0$ and $R_{ss} = \sqrt{4} = 2$.

Consider the initial condition $R(0) = 1.0$. We can predict the change in the radius from figure 1.

- $r = -4 \quad R(0) = 1 \quad \dot{R} < 0 \quad \Rightarrow t \to \infty \;\; R(t) \to 0$

  Origin (0,0) is a **stable fixed point**.

- $r = 0 \quad R(0) = 1 \quad \dot{R} < 0 \quad \Rightarrow t \to \infty \;\; R(t) \to 0$

  Origin (0,0) is a **stable fixed point**.

- $r = +4 \quad R(0) = 1 \quad \dot{R} > 0 \quad \Rightarrow t \to \infty \;\; R(t) \to 2$

  Origin (0,0) is an **unstable fixed point**.

  The steady-state radius is $R_{ss} = 2$ where $\dot{R} = 0$

Next, we can solve the system equations to give the time evolution of the trajectories given the initial conditions. The solution of the θ subsystem is

$$\theta = \omega t$$

The R subsystem is solved using the Python function **odeint**. The trajectories when $r < 0$ for the R and θ subsystems are shown in figure 1. The more negative the r value then the greater the rate of the decay for R reach zero. Figure 2 shows the time evolution of the trajectories when $r > 0$. For positive r values, the motion becomes sinusoidal with a period $T = 1$ when $\omega = 2\pi$ and the amplitude of the oscillation is $\sqrt{r}$.

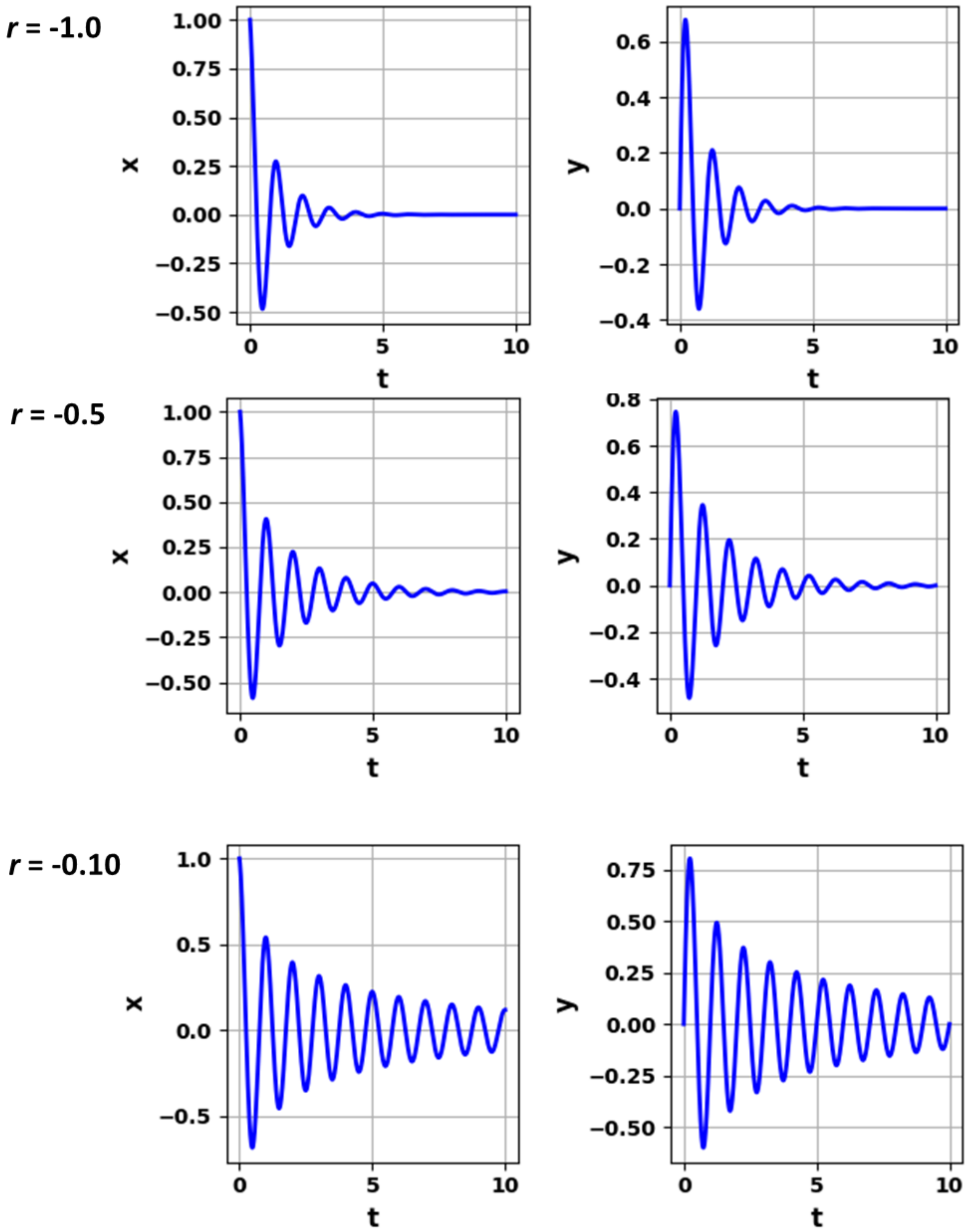

Fig. 1 The more negative the value of the control parameter *r*, the more rapidly the oscillation decay to zero.

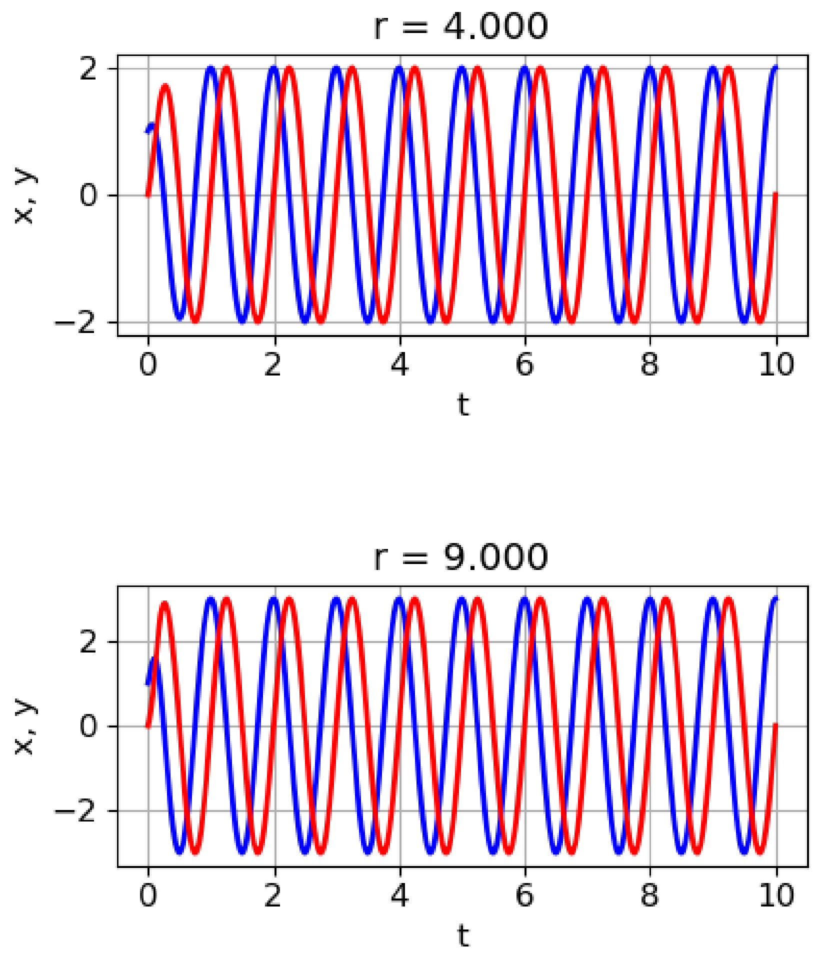

Fig. 2. Time evolution of the system $r > 0$. The trajectories (x red, y blue are sinusoidal with period $T = 1.00$ and amplitude $\sqrt{r}$.

Once, the ODEs have been solved, the phase portrait can be plotted for different r values and different initial conditions $\left(R(0), \theta(0)\right)$.

Figure 3 shows the phase portraits for different negative r values. The less negative the r value than the more slowly the trajectories spiral towards the fixed point at the Origin (0, 0).

$r < 0 \quad f'(0) = r < 0$ Origin (0, 0) is a **stable spiral**

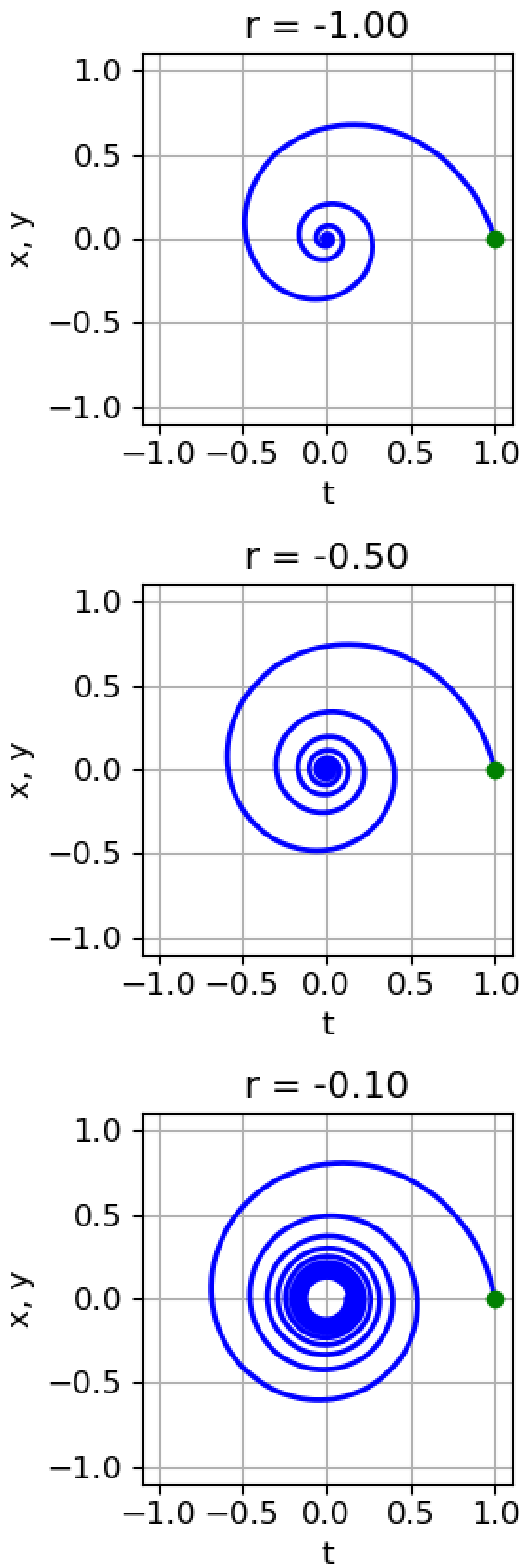

Fig. 3A. Phase portraits ($r < 0$). The more negative the value of the control parameter r, the more rapidly the oscillation decay to fixed point at the Origin (0, 0).

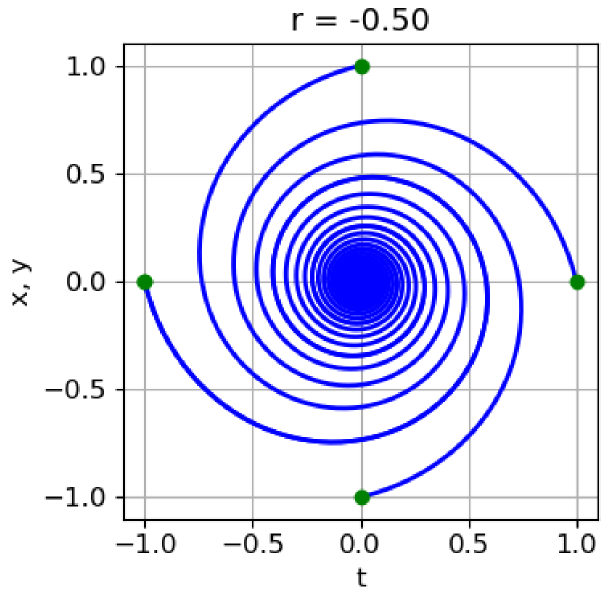

Fig. 3B. Phase portrait of the system $r < 0$ with different initial conditions. The fixed point is at the Origin (0, 0) is a **stable spiral** and all trajectories are attracted to it in an anticlockwise direction.

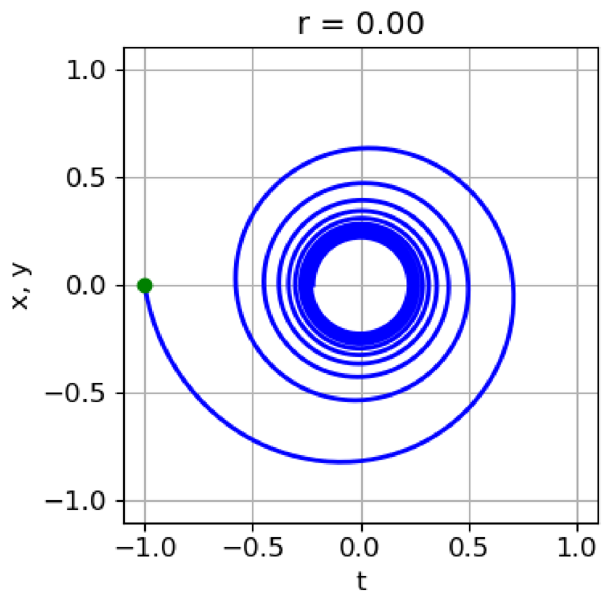

Fig. 3C. Phase portrait of the system $r = 0$. The fixed point is at the Origin (0, 0) is a **weak stable spiral** and all trajectories are attracted to it very slowly in an anticlockwise direction.

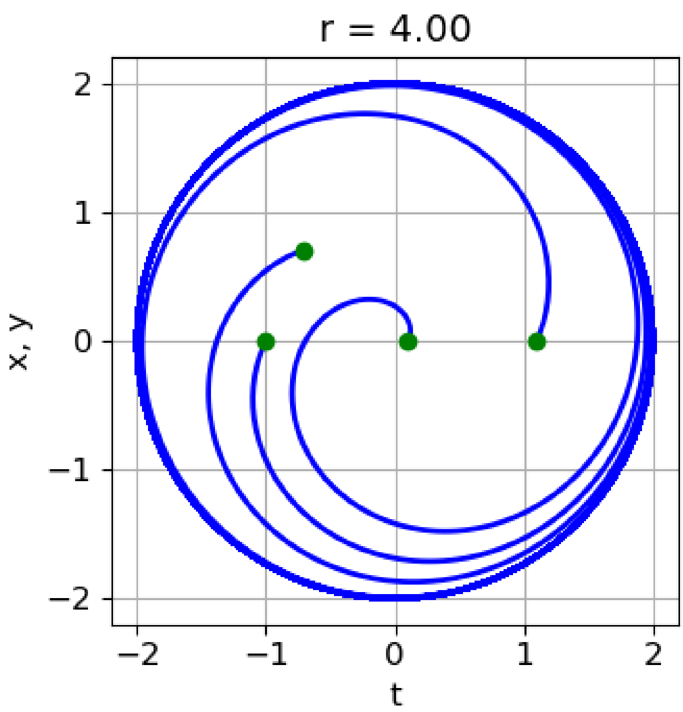

Fig. 4 Phase portrait plot showing the **limit cycle** with radius R

$$R_{ss} = \sqrt{r} \quad R_{ss} = \sqrt{4} = 2.00$$

When $r > 0$ all trajectories are repelled from the Origin which is now an **unstable** fixed point and the **attractor** is a **stable circular limit cycle** with radius $R_{ss} = \sqrt{r}$. In this example, in terms of the flow in phase space, the supercritical Hopf bifurcation occurs at $r = 0$ when the stable spiral changes into an unstable spiral surrounded by a circular orbit of radius $R_{ss} = \sqrt{r}$. When $r < 0$, the fixed point is at the Origin (0, 0) and is a stable spiral and all trajectories are attracted to it in anticlockwise direction. For $r = 0$, the Origin is still a stable spiral, but is very weak. For $r > 0$, the Origin is an unstable spiral, and the orbit in phase space is a stable limit cycle of radius $R = \sqrt{r}$.

The phase portraits can be shown in Python as a streamplot. Such plots help visualize a trajectory for different initial conditions without solving the system ODEs.

Figure 5 shows phase portraits and trajectories for $r = -4$, $r = 0$ and $r = 4$. For $r < 0$, the Origin (0, 0) acts as a stable fixed point and strongly attracts the trajectory. The trajectory is a stable spiral. For $r = 0$, the Origin (0, 0) acts as a weak stable fixed point and weaky attracts the trajectory. The trajectory is a stable spiral. When $r > 0$ then the dynamics is very different. A **bifurcation** occurs at $r = 0$ where the Origin changes from stable to unstable as r passes through zero and the spiral trajectory disappears and becomes a limit cycle about the Origin.

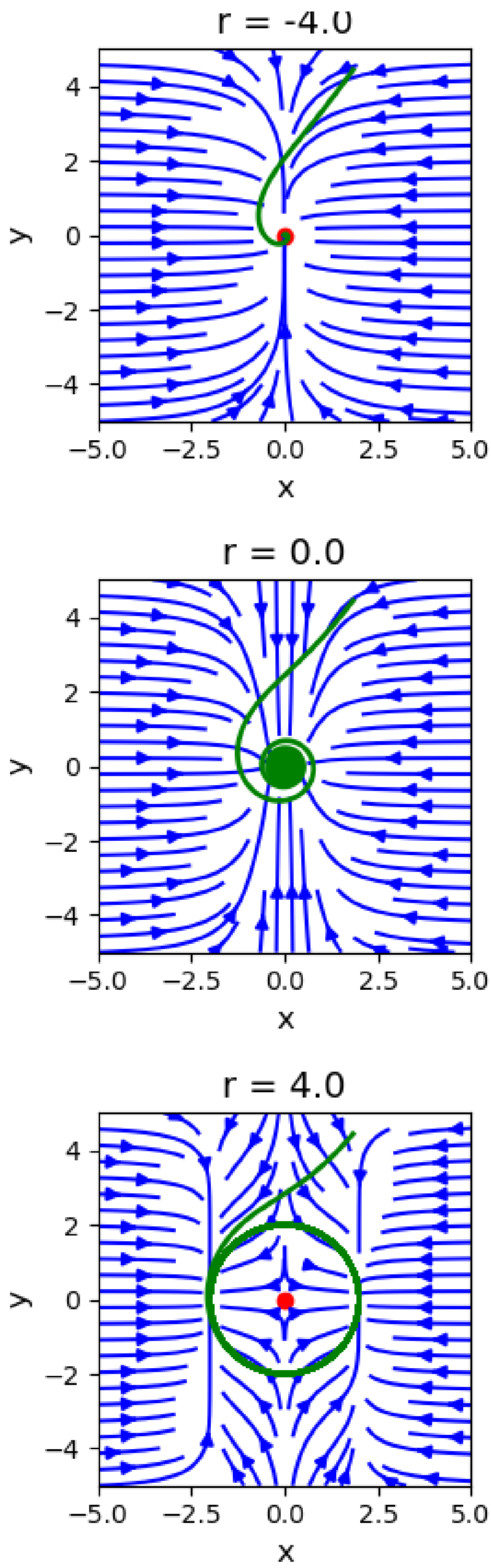

Fig. 5. Phase portrait and trajectory for $r = -4$, $r = 0$ and $r = 4$.

Mathematical analysis

The system equations are

$$\dot{R} = r\,R - R^3 \qquad \dot{\theta} = \omega$$

and need to be analysed for three cases: $r < 0$, $r = 0$ and $r > 0$.

The solution of the θ subsystem is

$$\theta = \omega t$$

Steady-state solutions for the R subsystem are determined from $\dot{R} = 0$ and the stability from $f(R) = r\,R - R^3 \quad f'(R) = r - 3R^2$.

One steady-state solution is when the radius vector is zero, $R = 0$ and this corresponds to the fixed point at the Origin (0, 0). The stability at the Origin depends on the control parameter r

$r < 0 \quad f'(0) = r < 0$ Origin is a **strong stable fixed point (spiral)**

$r = 0 \quad f'(0) = r = 0$ Origin is a **weak stable fixed point (spiral)**

$r > 0 \quad f'(0) = r > 0$ Origin is an **unstable fixed point (limit cycle)**

The steady-state solution for $r > 0$ is $R = \sqrt{r}$ and this corresponds to a circle of radius $R = \sqrt{r}$. So, for $r \le 0$ the Origin is the only fixed point and for $r > 0$ fixed points exist at on the circle of radius $R = \sqrt{r}$. The stability is stable since $f'(\sqrt{r}) = -2r < 0$ and the flow will be attracted to this circle.

The Cartesian form of our system equations is

$$\dot{x} = r\,x - y - x\left(x^2 + y^2\right) \quad \dot{y} = x + r\,y - y\left(x^2 + y^2\right)$$

where $\omega = 1$.

We can calculate the Jacobian for the fixed point at the Origin (0, 0).

$$f = r\,x - y - x\left(x^2 + y^2\right)$$

$$\partial f / \partial x = r - 3x^2 - y^2 \quad \partial f / \partial y = -1 - 2\,x\,y$$

$$g = x + r\,y - y\left(x^2 + y^2\right)$$

$$\partial g / \partial x = 1 - 2\,x\,y \quad \partial g / \partial y = r - x^2 - 3y^2$$

$$\mathbf{J}(0,0) = \begin{pmatrix} r & 1 \\ -1 & r \end{pmatrix}$$

The eigenvalues for the Jacobian matrix at the Origin (0, 0) are

$$eigenvalues = \left(r + j,\, r - j\right) \quad j = \sqrt{-1}$$

$$r = -4 \;\; \lambda_0 = -4 + j \;\; \lambda_1 = -4 - j \qquad r = +4 \;\; \lambda_0 = +4 + j \;\; \lambda_1 = +4 - j$$

The bifurcation point (0, 0) is called a focus or spiral point when eigenvalues are complex-conjugate. The focus is stable when the eigenvalues have negative real part and unstable when they have positive real part. Therefore, the Origin is a stable spiral when $r < 0$ and an unstable spiral when $r > 0$. The eigenvalues cross the imaginary axis from left to right as the parameter r changes from negative to positive values. Hence, a **supercritical Hopf bifurcation** occurs when a stable spiral changes into an unstable spiral surrounded by a limit cycle.