DOING PHYSICS WITH PYTHON

[2D] NON-LINEAR DYNAMICAL SYSTEMS

SUBCRITICAL HOPF BIFURCATIONS

Ian Cooper
matlabvisualphysics@gmail.com

DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive **GitHub**

ds1510.py

SUBCRITICAL HOPF BIFURCATION

The Hopf bifurcations come in both super- and subcritical varieties. The subcritical case is always much more dramatic, and potentially dangerous in engineering applications. After the bifurcation, the trajectories must jump to a distant attractor, which may be a fixed point, another limit cycle, or infinity.

Consider a [2D] system with bifurcation parameter r

$$\dot{x} = r\,x - y - x\left(x^2 + y^2\right) - x\left(x^2 + y^2\right)^2$$

$$\dot{y} = x + r\,y + y\left(x^2 + y^2\right) - y\left(x^2 + y^2\right)^2$$

The system has a unique fixed point at the Origin (0, 0).

The ODEs governing the system are best expressed in polar coordinates where

$$x = R\cos\theta \quad y = R\sin\theta \quad R^2 = x^2 + y^2 \quad \tan\theta = y/x$$

After some tedious algebra, the ODEs in polar coordinates are

$$\dot{R} = r\,R + R^3 - R^5 \quad \dot{\theta} = 1$$

The ODEs now are decoupled and are easy to analyse for $r < 0$, $r = 0$ and $r > 0$.

The important difference from the earlier supercritical case is that the cubic term R^3 is now destabilizing; it helps to drive trajectories away from the Origin.

Mathematical analysis

The fixed points R_e are determined from $\dot{R} = 0$ and the stability from

$$\dot{R} = 0 \;\Rightarrow\; R_e^{\,4} - R_e^{\,2} - r = 0$$

(1) $$R_e^{\,2} = \frac{1 \pm \sqrt{1+4r}}{2}$$

$$f(R) = r\,R + R^3 - R^5 \quad f'(R_e) = r + 3R_e^{\,2} - 5R_e^{\,4}$$

$r < -1/4$ Origin (0, 0) is the only fixed point and is a stable spiral.

$-1/4 < r < 0$ Three steady-states

Consider the case when r = -0.2 then from equation 1, the three steady-states and their stability are:

$$R_e = 0 \quad f'(0) = r = -0.2 < 0 \quad \text{stable fixed point}$$

$$R_e = 0.851 \quad f'(0.851) = -0.647 < 0 \quad \text{stable radius}$$

$$R_e = 0.526 \quad f'(0.526) = 0.247 > 0 \quad \text{unstable radius}$$

$r = 0$ Two steady-states

$$R_e = 0 \quad f'(0) = r = 0 \quad \text{unstable fixed point}$$

$$R_e = 1 \quad f'(0) = -2 < 0 \quad \text{stable radius}$$

$r > 0$ Two steady-states

Consider the case when r = +0.2 then from equation 1, the two steady-states and their stability are:

$$R_e = 0 \quad f'(0) = r = 0.2 > 0 \quad \text{unstable}$$

$$R_e = 1.082 \quad f'(1.082) = -3.142 < 0 \quad \text{stable}$$

The steady-state solutions are: the Origin (0, 0) which is unstable and all points on the circle radius R_e are stable. This gives a stable spiral in the phase portrait.

Graphical analysis

A good starting point to investigate the dynamics of the system is to plot the system equation $(R, \dot{R})$ as shown in figure 1. In the analysis of the system dynamics, you need to consider the four cases $r < -0.25$, $-0.25 < r < 0$ and $r > 0$.

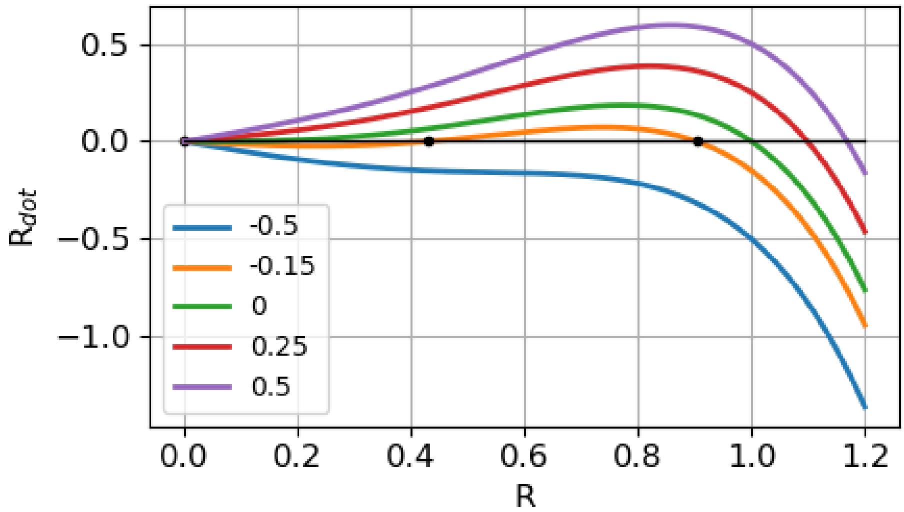


Fig. 1. The R subsystem plots for different r values. A positive slope indicates an instability and a negative slope indicates stability of the radius vector R.

r = -0.50 < -0.25 one stable fixed point, Origin (0, 0)

$R_e = 0$ stable (negative slope)

-0.25 < r = -0.15 < 0 three steady-state radii

$R_e = 0$ stable (negative slope)

$R_e = 0.429$ unstable (positive slope)

$R_e = 0.903$ stable (negative slope)

$r = 0$ two steady-state radii

$R_e = 0$ unstable (positive slope)

$R_e = 1.000$ stable (negative slope)

$r > 0$ two steady-state radii

$R_e = 0$ unstable (positive slope)

$r = 0.25$ $R_e = 1.099$ stable (negative slope)

$r = 0.50$ $R_e = 1.169$ stable (negative slope)

Phase portraits and trajectories

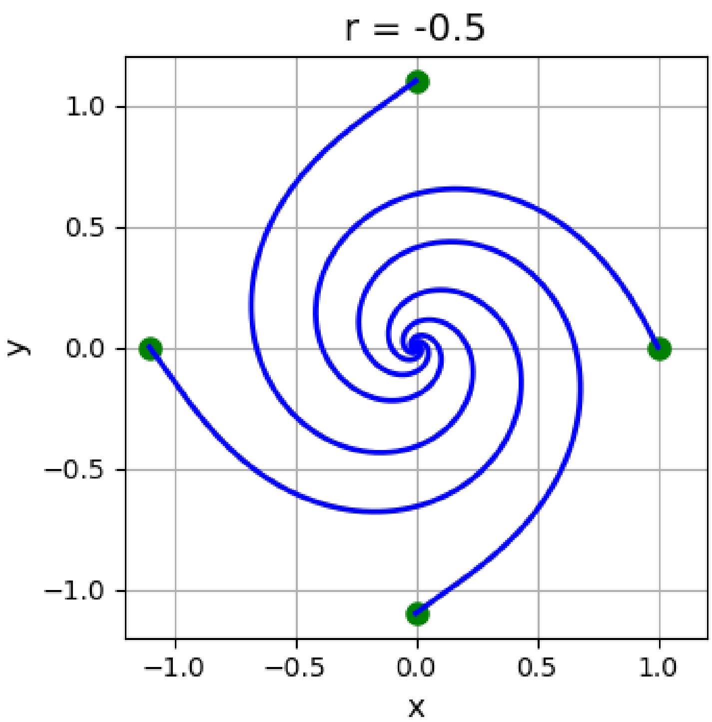


Fig. 2.1. **$r = -0.50 < -0.25$** The Origin (0, 0) is a stable focus. For all initial conditions where the flow spirals to the Origin.

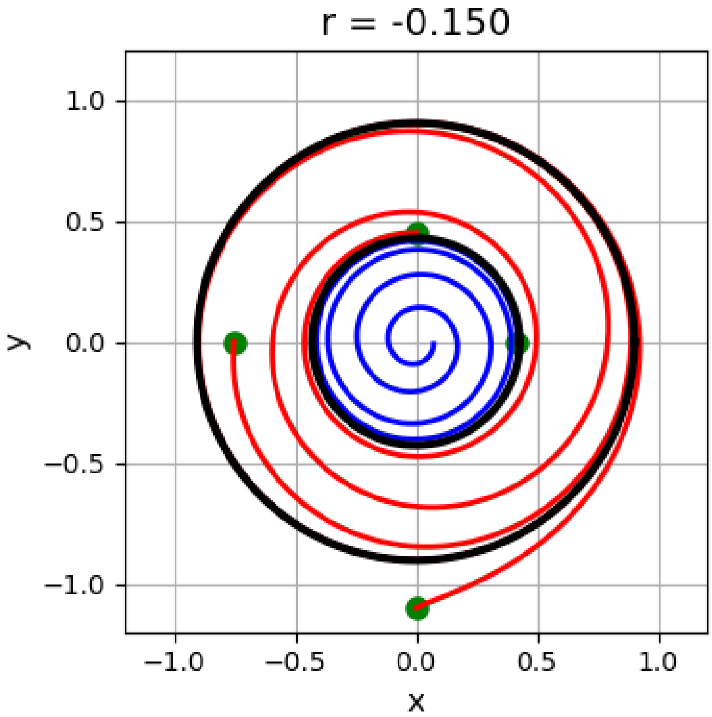


Fig. 2.2. **-0.25 < *r* = -0.150 < 0** The Origin (0, 0) is a stable focus and attracts all orbits for $R(0) < R_e = 0.429$ (unstable steady-state radius). For all initial conditions $R(0) > R_e = 0.429$, the orbits are attracted to the stable steady-state radius $R_e = 0.903$ and thus the steady-state orbit is a circular limit cycle. The bifurcation occurs at the bifurcation parameter $r = r_C = -0.25$.

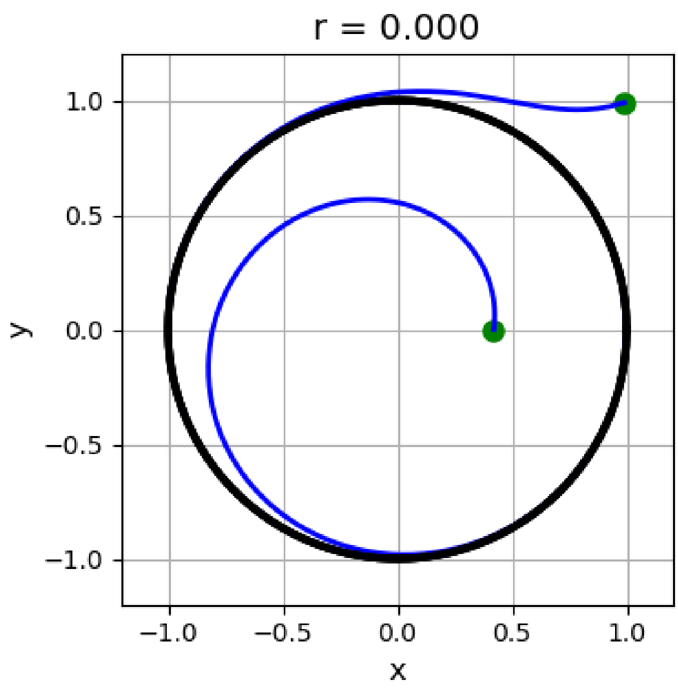


Fig. 2.3. The Origin (0, 0) is **unstable**. The circle with radius $R_e = 1$ is a **stable** limit cycle. For all initial conditions other than the Origin are attracted to the steady-state circle with radius $R_e = 1$.

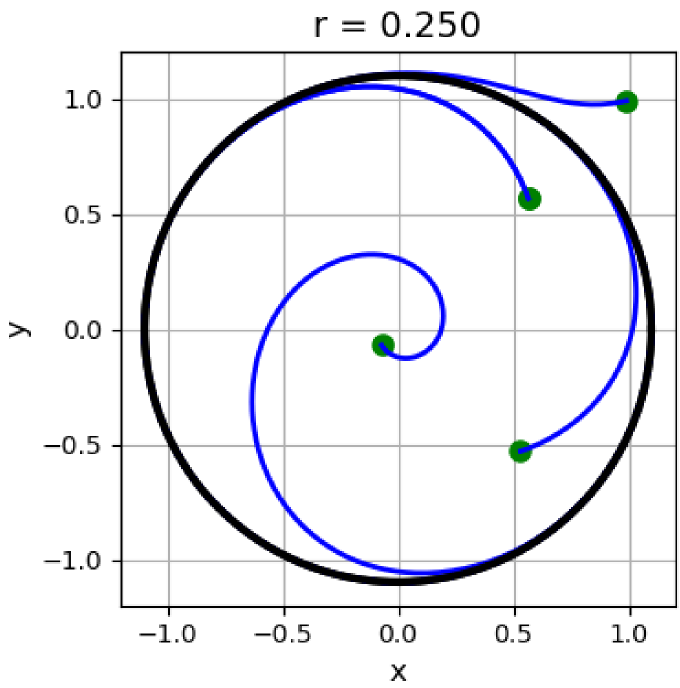
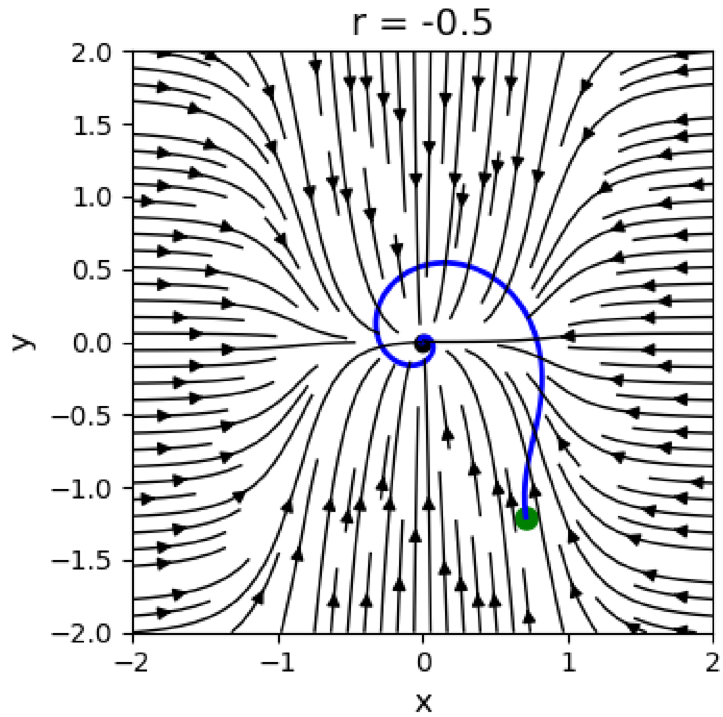


Fig. 2.3. The Origin (0, 0) is **unstable**. The circle with radius $R_e = 1.099$ is a **stable** limit cycle. For all initial conditions other than the Origin are attracted to the steady-state circle with radius $R_e = 1.099$.

Phase portrait (streamplot), trajectory, and time evolution

r **< -0.25**



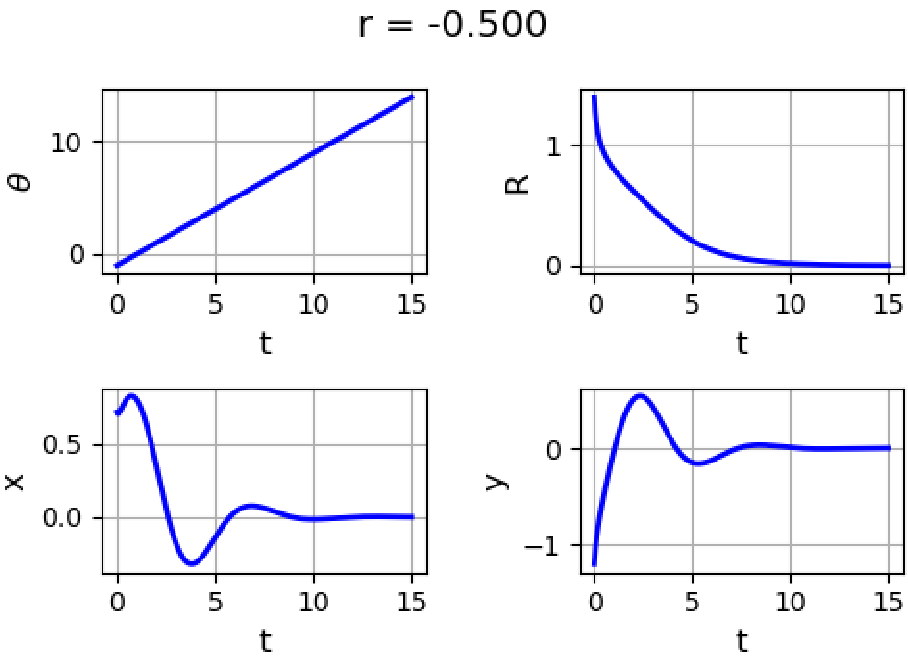


Fig. 3.1. If ***r*** **< -1/4**, then the only fixed point is the Origin (0, 0) and the all orbits are **stable spirals**.

$-1/4 < r < 0$ there are three steady-states.

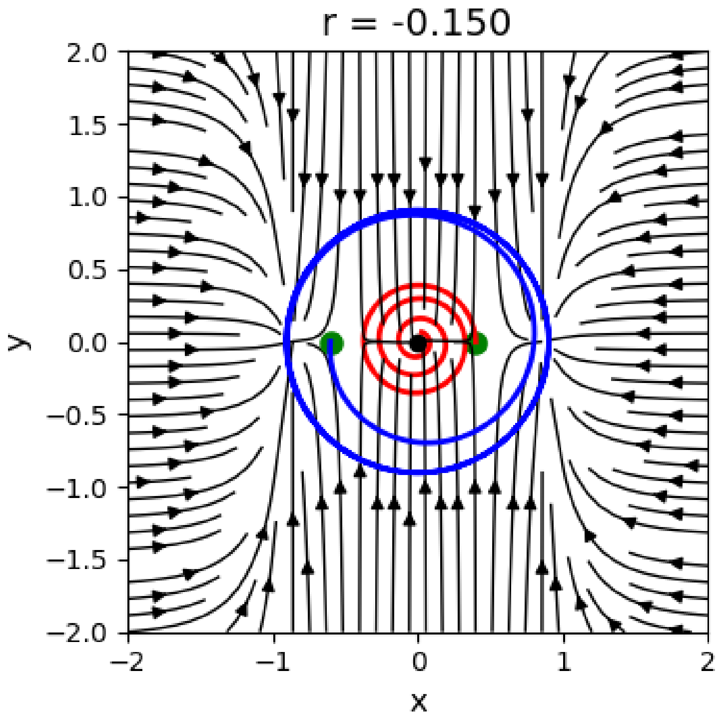


Fig. 4.1 $R(0) = 0.60$ trajectory spirals to the stable steady-state radius $R_e = 0.903$ and is repelled from the unstable steady-state radius $R_{ss} = 0.429$. $R(0) = 0.40$ trajectory spirals to the fixed point at the Origin and is repelled from the unstable steady-state radius $R_{ss} = 0.429$.

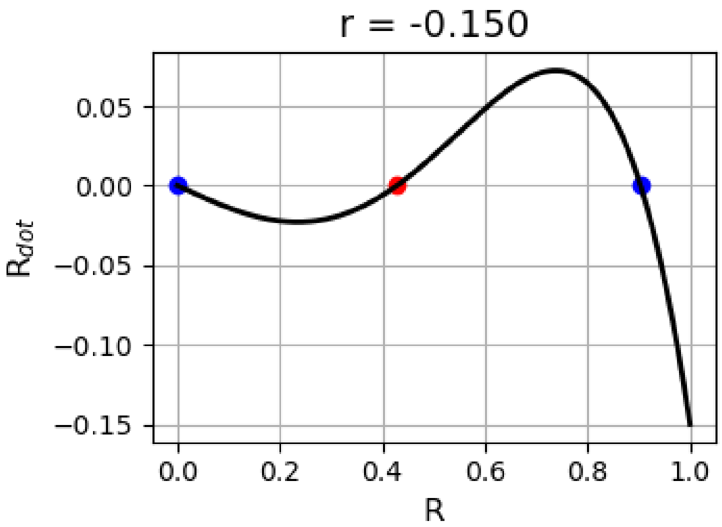


Fig. 4.2. The Origin ($R = 0$) is a stable fixed point (negative slope). The steady-state radius $R_{ss} = 0.429$ is unstable (positive slope). The steady-state radius $R_{ss} = 0.903$ is stable (negative slope).

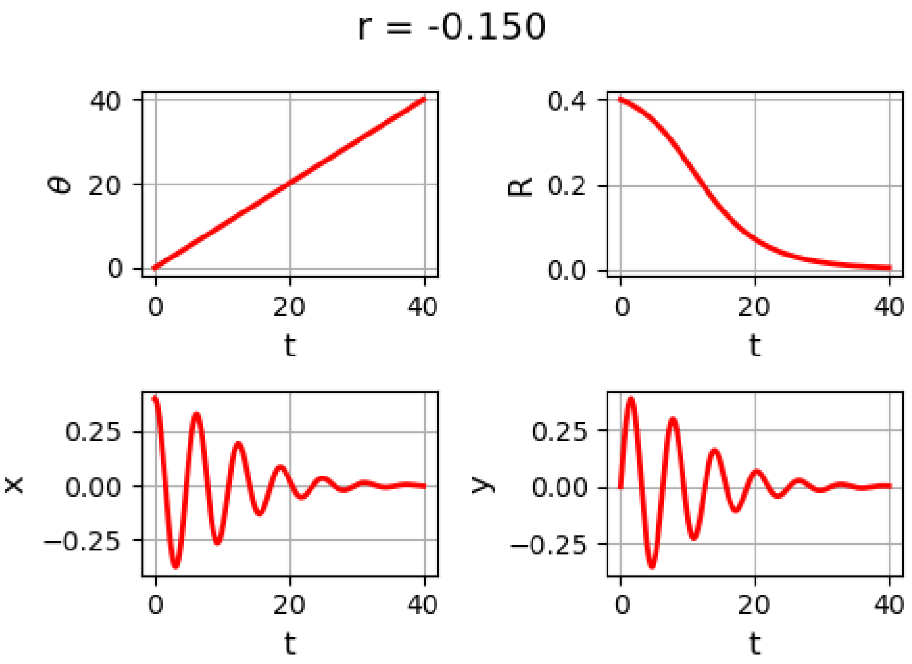


Fig. 4.3. $R(0) = 0.40$. Trajectory spirals to the fixed point at the Origin.

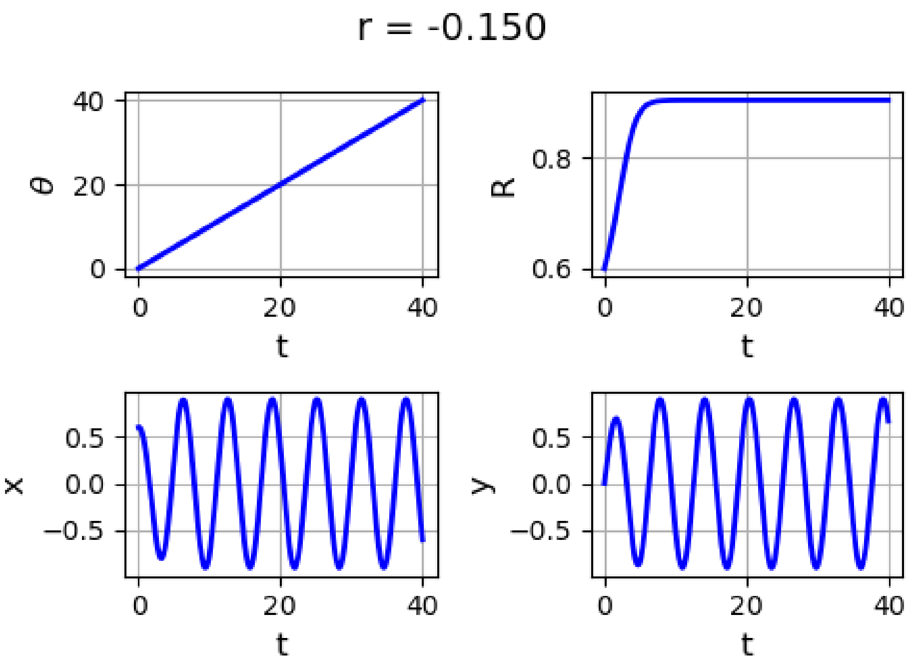


Fig. 4.4. $R(0) = 0.60$. Trajectory spirals to the stable steady-state radius $R_{SS} = 0.903$.

For -**0.25 < *r* < 0** there are two attractors, a stable limit cycle and a stable fixed point at the Origin. Between them lies an unstable cycle. For initial conditions near the unstable steady-state radius are repelled to either the fixed point at the Origin or the stable steady-state radius. This makes it impossible to make a prediction on the trajectory for small changes in the initial conditions near the unstable steady-state radius.

r = 0

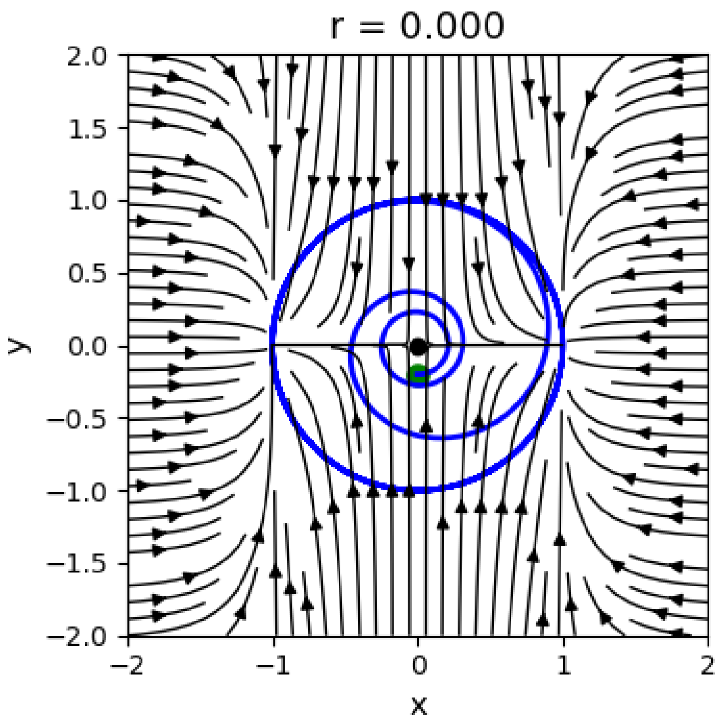


Fig. 5.1. The unstable steady-state radius has disappeared when *r* is increased to zero and the stability of the Origin has changed from stable to unstable. For initial conditions near the Origin, all trajectories spiral outward as they are attracted to the steady-state radius $R_{SS} = 1$.

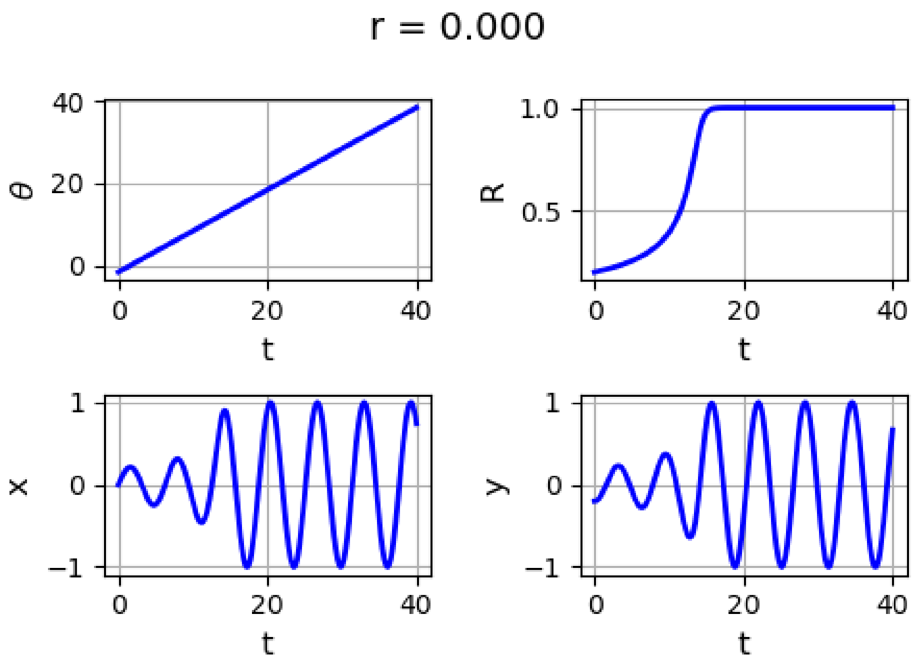


Fig. 5.2. The radius R of the orbit increases to its steady-state value $R_{SS} = 1.00$ and the orbit is a circle centred on the Origin.

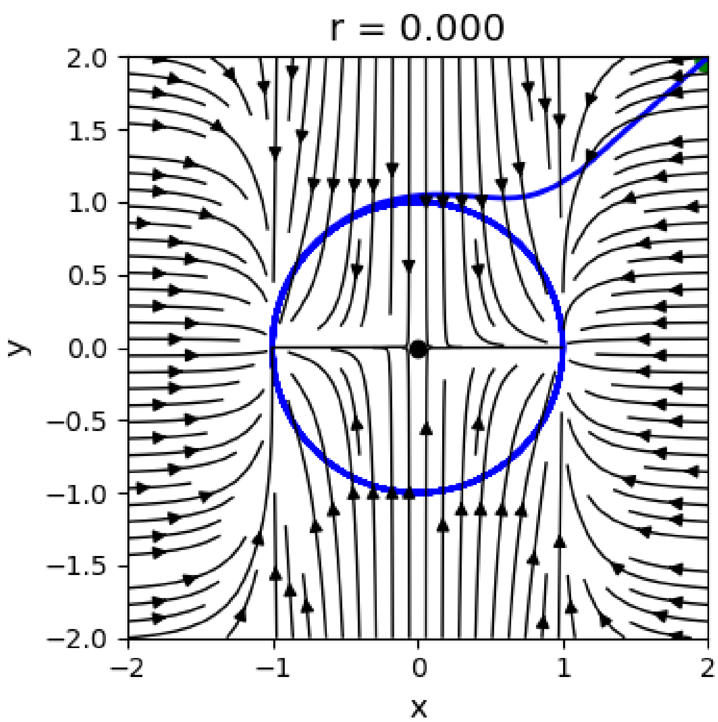


Fig. 5.3. For all initial conditions where $R(0) > R_{SS} = 1.00$ converge to the circle of radius $R_{SS} = 1.00$.

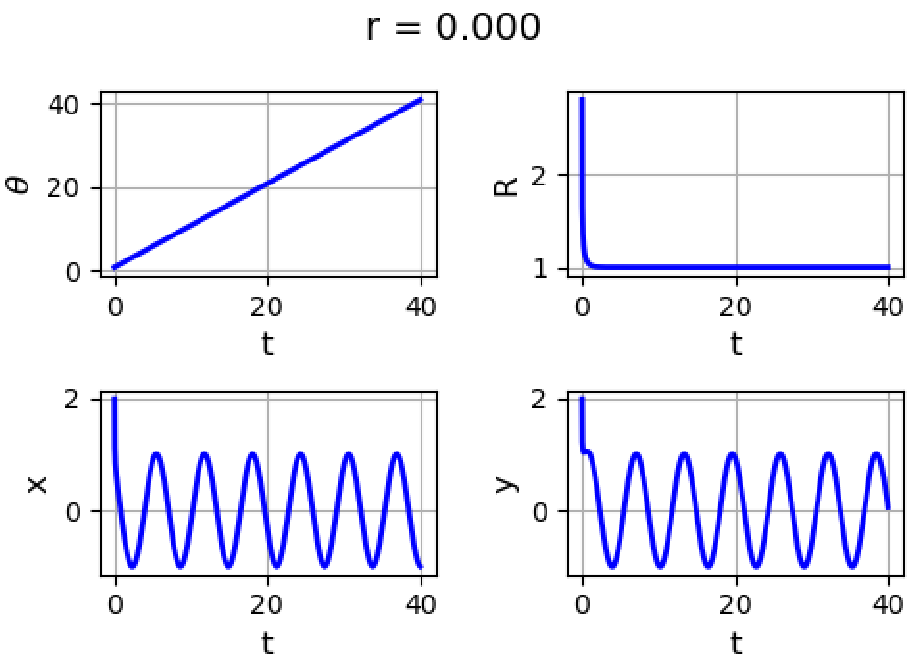


Fig. 5.4. For all initial conditions where $R(0) > R_{SS} = 1.00$ converge to the circle of radius $R_{SS} = 1.00$. The trajectories are now a circle of radius $R_{SS} = 1.00$ centred on the Origin.

***r* = 0** is a **bifurcation point**. As r increases, the unstable cycle tightens like a noose around the fixed point (0, 0). A **subcritical Hopf bifurcation occurs at *r* = 0**, where the unstable cycle shrinks to zero amplitude and engulfs the Origin, rendering it unstable.

$r > 0$

There are two fixed points when $r > 0$, (0, 0) which is unstable and a steady-state radius.

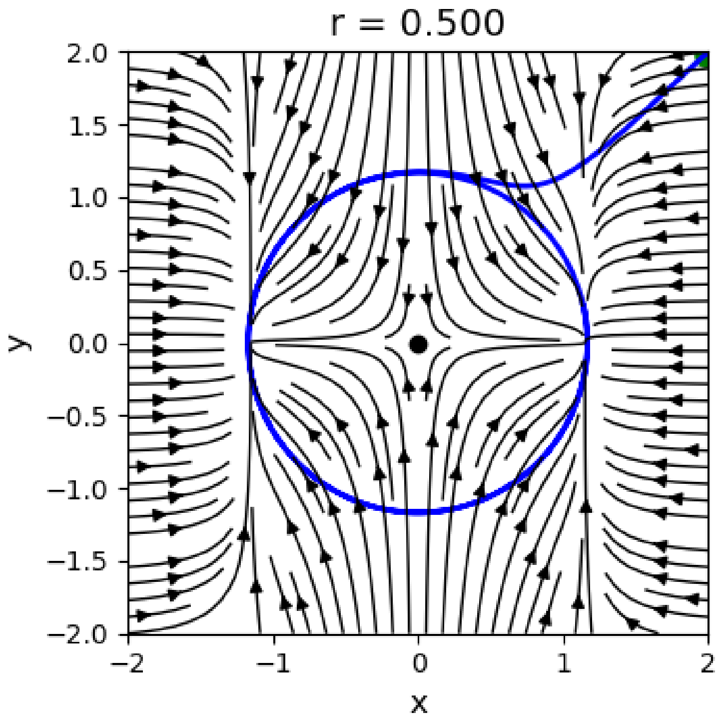


Fig. 6.1. All trajectories will be draw to the stable limit cycle.

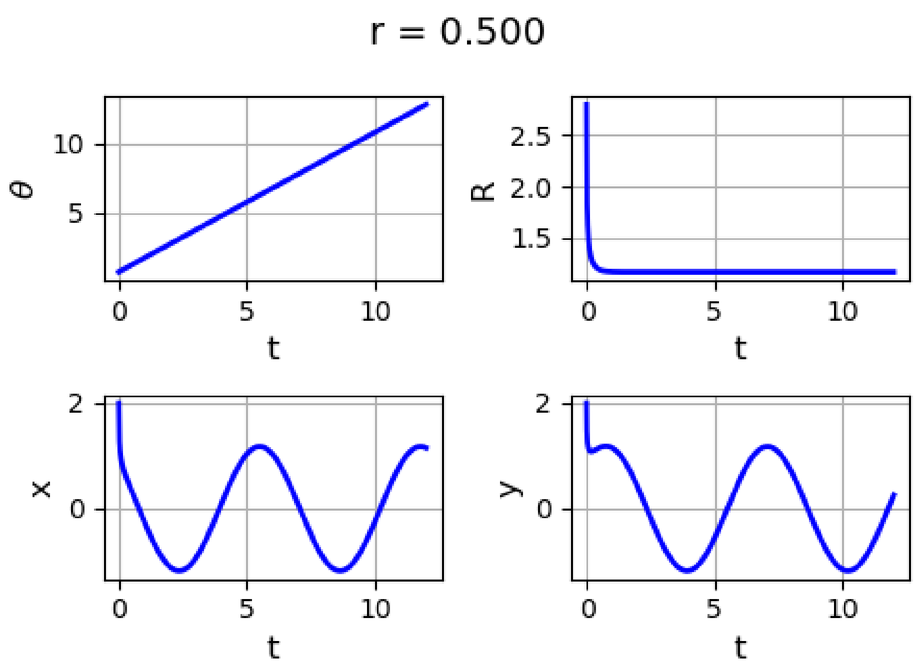


Fig. 6.2. The trajectory is strongly drawn to the limit circle of radius R_{SS} = 1.169 with large amplitude oscillations.

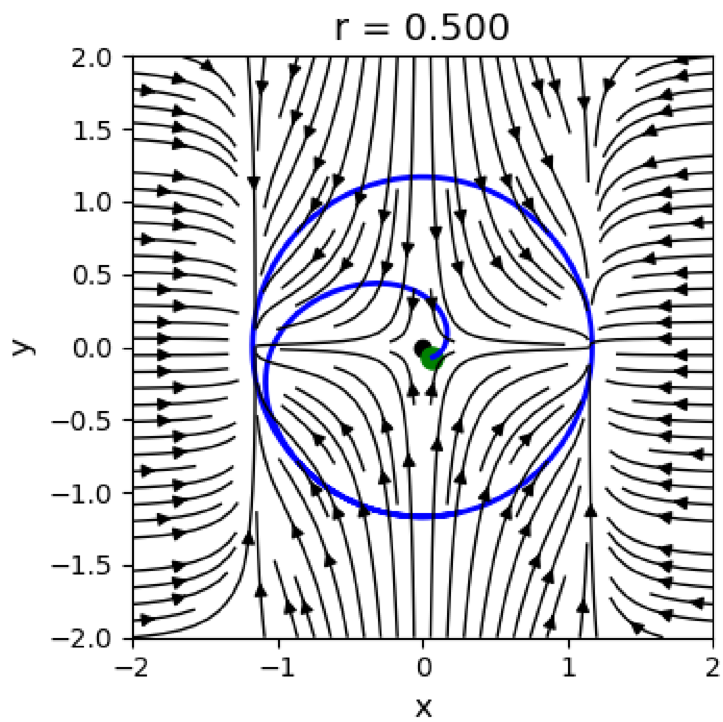


Fig. 6.3. The trajectory is repelled from the Origin and attracted to the steady-state circular orbit of radius RSS = 1.169.

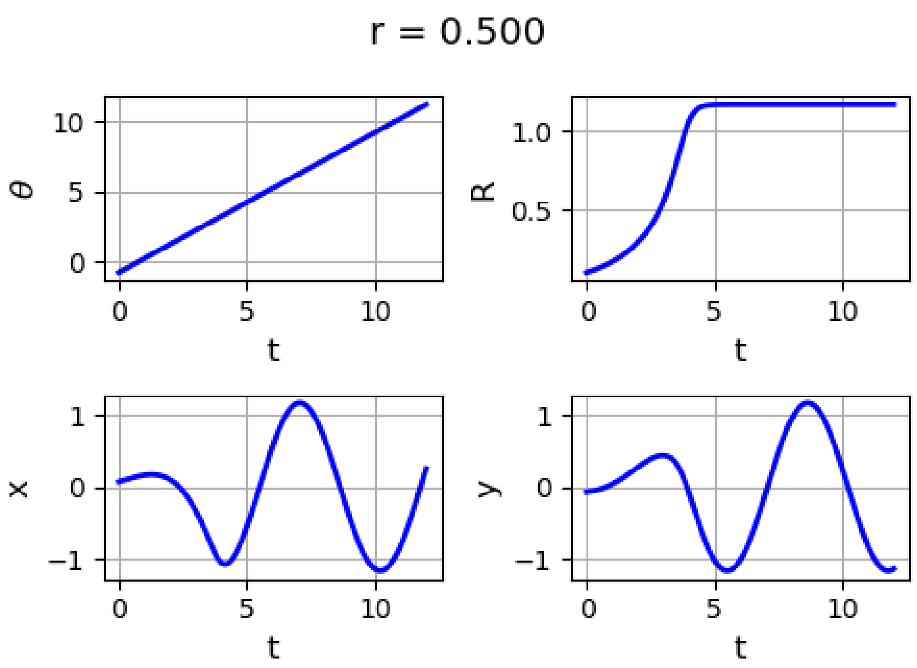


Fig. 6.4. ***r* > 0** the large-amplitude limit cycle is now the only attractor. Solutions that used to remain near the Origin are now forced to grow into large-amplitude oscillations.

A **subcritical Hopf bifurcation** is a dynamic event where a stable equilibrium point in a system abruptly loses its stability and becomes unstable. This transition occurs when an unstable, pre-existing limit cycle (a solution that oscillates periodically) shrinks and collides with the equilibrium point, transferring its instability. Unlike a supercritical Hopf bifurcation, the resulting stable periodic solution is a large-amplitude oscillation that is not born continuously from zero but jumps into existence, often leading to hard, dangerous, or discontinuous changes in behaviour, known as a "hard" bifurcation