DOING PHYSICS WITH PYTHON

[2D] NON-LINEAR DYNAMICAL SYSTEMS

RABBITS AND SHEEP POPULATION DYNAMICS

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ds1700.py

For [2D] dynamical systems, there are five generic types of fixed points. They are classified according to the eigenvalues of the linearized dynamics Jacobian matrix at the fixed point. For a real Jacobian 2 × 2 matrix, the eigenvalues must be real or else must be a complex conjugate pair. The five types of fixed points are then

Unstable node $\lambda_1 > 0 \quad \lambda_2 > 0$

Saddle point $\lambda_1 > 0 \quad \lambda_2 < 0$

Stable node $\lambda_1 < 0 \quad \lambda_2 < 0$

Unstable spiral $\mathrm{Re}(\lambda_1) > 0 \quad \lambda_2 = \lambda_1^*$

Stable spiral $\mathrm{Re}(\lambda_1) < 0 \quad \lambda_2 = \lambda_1^*$

We can model the interactions between rabbits (x) and sheep (y) as they compute for limited resources (grass).

The system equations are

rabbits $\dot{x} = x(3 - x - 2y)$

sheep $\dot{y} = y(2 - x - y)$

The competition for food is given by the terms $x\,y$. When either population x or y vanishes, the remaining population is governed by the logistic equation, where the population will flow to a nonzero fixed point.

rabbits: $y = 0$ $\dot{x} = \frac{1}{3}x(1 - \frac{x}{3})$

growth rate $r = 1/3$ carrying capacity $K = 3$

sheep: $x = 0$ $\dot{y} = \frac{1}{2}y(1 - \frac{y}{2})$

growth rate $r = 1/2$ carrying capacity $K = 2$

Nullclines

rabbits: $\dot{x} = x(3 - x - 2y) = 0$ $x = 0$ $y = (3 - x)/2 = 0$

sheep: $\dot{y} = y(2 - x - y) = 0$ $y = 0$ $y = 2 - x$

Fixed points: intersection of nullclines with each other and the axes $x = 0$ and $y = 0$. There are four fixed points.

- Origin (0, 0)
- $x = 0, y = 2$ (0, 2)
- $x = 3, y = 0$ (3, 0)
- $y = 2 - x = (3 - x)/2 \Rightarrow x = 1 \quad y = 1 \quad (1, 1)$

Jacobian matrix evaluated at the fixed point (x_e, y_e):

$$\mathbf{J}(x_e, y_e) = \begin{pmatrix} \partial f / \partial x & \partial f / \partial y \\ \partial g / \partial x & \partial g / \partial y \end{pmatrix}_{|x=x_e, y=y_e}$$

and has two eigenvalues, which are either both real or complex-conjugates. The eigenvalues λ and eigenvectors $\mathbf{V}$ are calculated using the Python function **eig**.

$$\mathbf{J}(x, y) = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}$$

$$\mathbf{J}(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

$\lambda_1 = 3 \quad \mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \lambda_2 = 2 \quad \mathbf{V}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ **unstable node**

$$\mathbf{J}(0,2) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$$

$$\lambda_1 = -2 \quad \mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \lambda_2 = -1 \quad \mathbf{V}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$ **stable node**

$$\mathbf{J}(3,0) = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$$

$$\lambda_1 = -3 \quad \mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \lambda_2 = -1 \quad \mathbf{V}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$ **stable node**

$$\mathbf{J}(1,1) = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

$$\lambda_1 = 0.414 \quad \mathbf{V}_1 = \begin{pmatrix} 1.414 \\ -1 \end{pmatrix} \qquad \lambda_2 = -2.414 \quad \mathbf{V}_2 = \begin{pmatrix} 1.414 \\ 1 \end{pmatrix}$$ **saddle**

The fast eigenvector direction is for the largest magnitude eigenvalue and the slow eigenvector direction is the slow eigenvector direction.

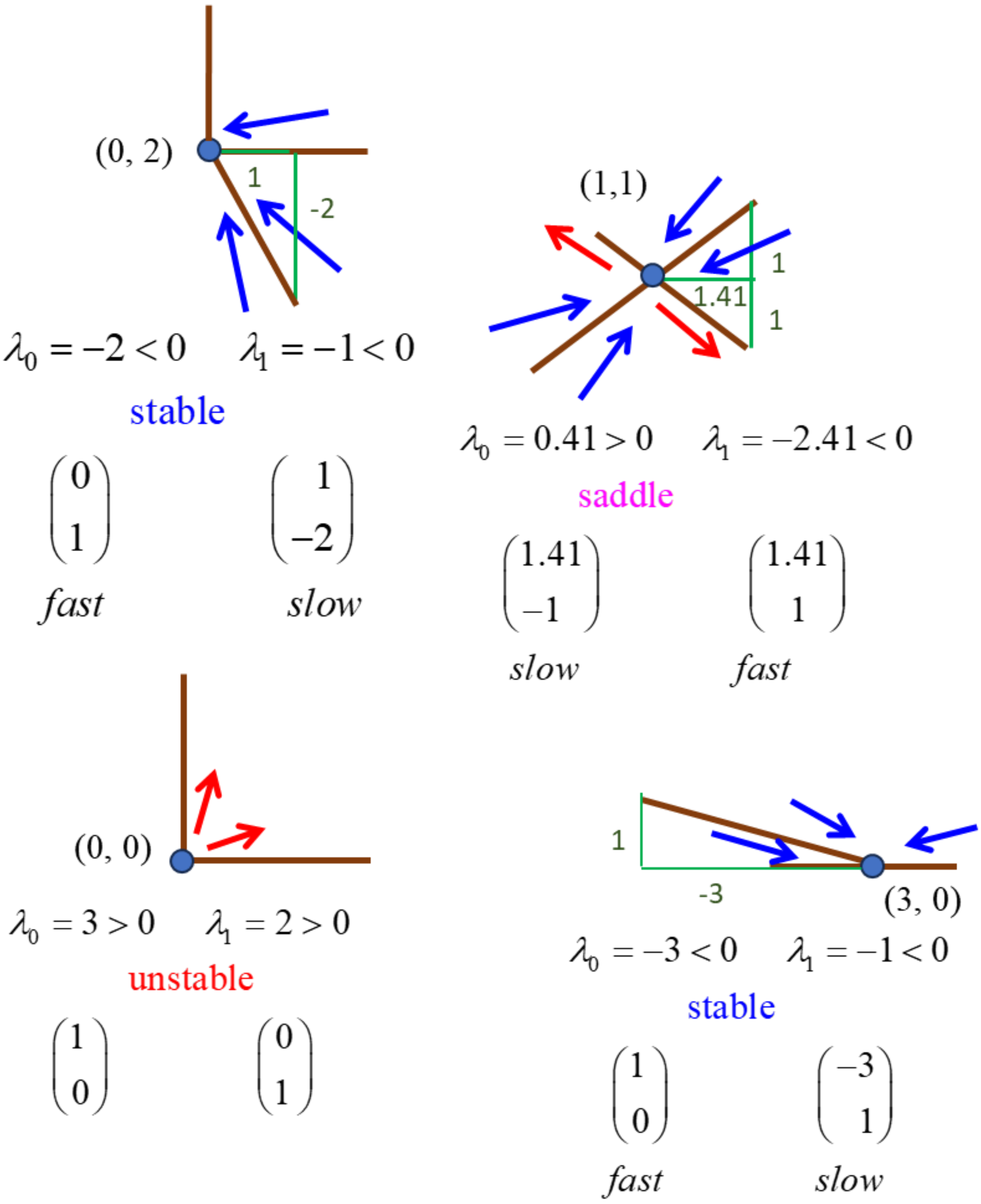

Fig. 1. Understanding the flow in the locality of each fixed point from the information provided by the eigenvalues and eigenvectors. The manifolds or eigenvector directions near the fixed points are shown in brown. At a **stable** fixed point, the flow is attracted to the fixed point, while it is repelled from an **unstable** fixed point. The flow near a fixed point is essentially parallel to the manifold (trajectories approach the slow eigenvector direction). For the **saddle point** (1, 1) you have trajectories which are repelled (unstable manifold) from the fixed point, while other trajectories are attracted (stable manifold).

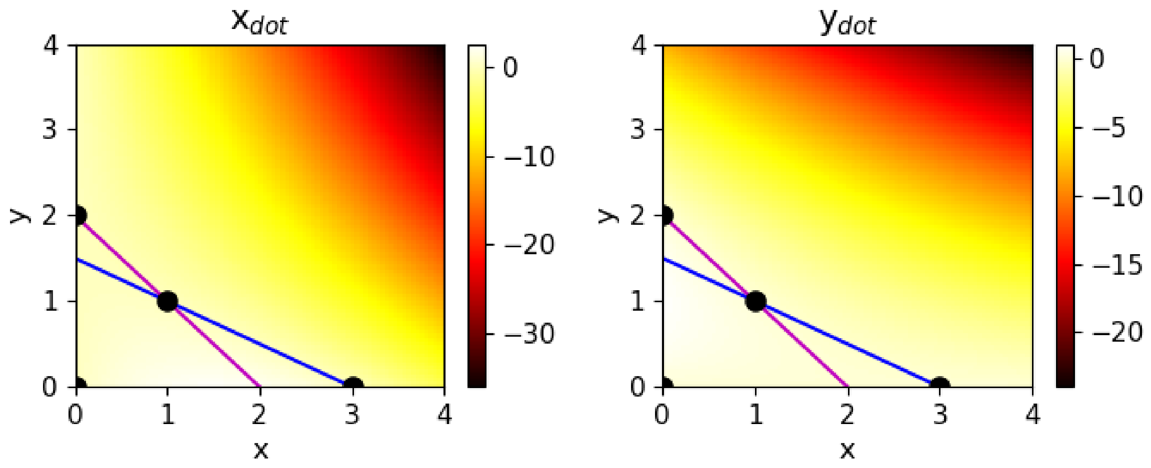

Fig. 2. The flow around the **fixed points** can be seen by examining the flow in the *x* and the *y* directions. The straight lines are the ***x*** and ***y*** nullclines.

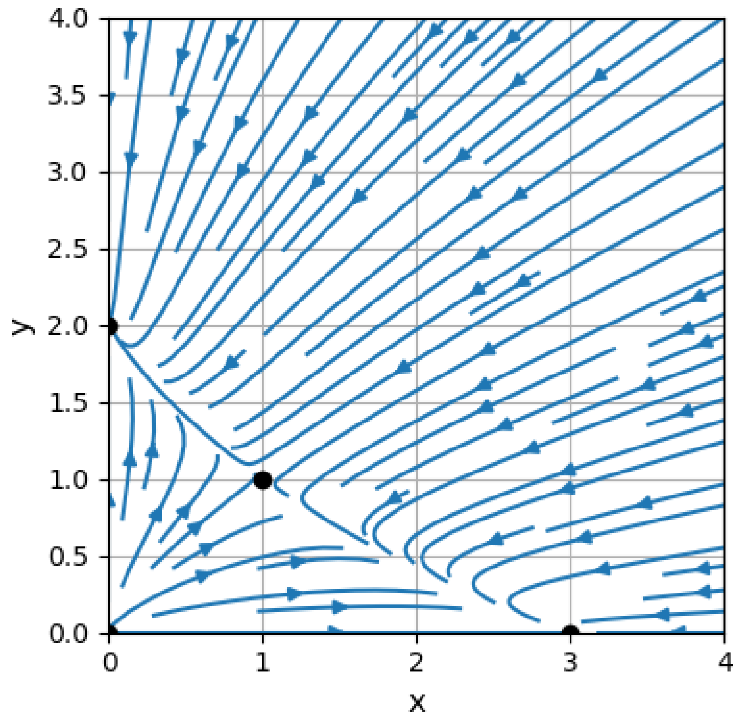

Fig. 3. Phase portrait as a streamplot. The flow is away from the fixed point at the Origin (0, 0) and attracted to the fixed points (3, 0) and (0, 2). The saddle point (1, 1) attracts the flow far from the saddle but strongly repels the flow in the local region near the saddle.

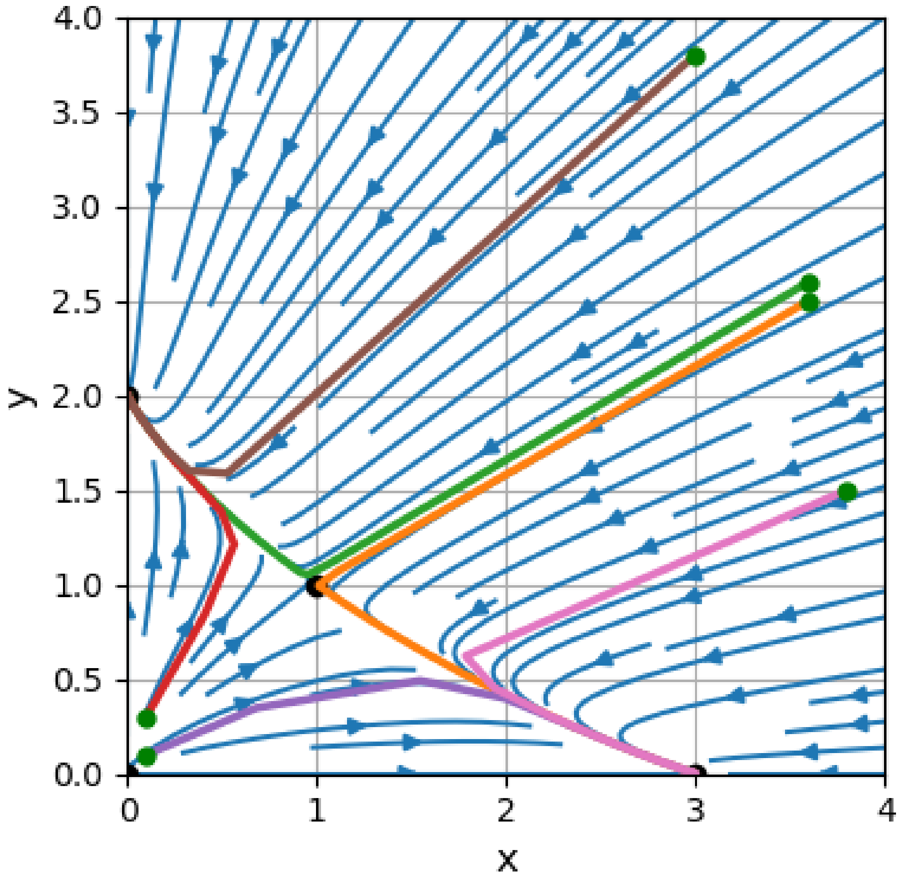

Fig. 4. Phase portrait and trajectories with different initial conditions. The phase space is divided into a number of basins of attraction. Slight differences in initial conditions may lead to very different trajectories. For the saddle point (1, 1) the flow is in along the stable manifold and out along the unstable manifold. The manifolds act as a separatrix (like a big fault line separating trajectories).

REFERENCES

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