DOING PHYSICS WITH PYTHON

[2D] NON-LINEAR DYNAMICAL SYSTEMS

CONSERVATIVE SYSTEMS

DOUBLE POTENTIAL WELL

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ds1800.py

We can consider the motion of a particle of mass m whose motion is governed by a potential energy function V corresponding to a double potential well. The potential energy for the double well is

$$V(x) = -\tfrac{1}{2}x^2 + \tfrac{1}{4}x^4$$

Newton's second law can be expressed as

$$m\ddot{x} = -\frac{dV}{dx}$$

$$\ddot{x} = \frac{x - x^3}{m}$$

This equation can be written as a system of first order differential equations ($m = 1$)

$$\dot{x} = y$$
$$\dot{y} = x - x^3$$

A good starting point in the analysis of the system's dynamics is to plot the potential energy function $V(x)$ as shown in figure 1. The equilibrium positions occur when the force acting on the particle is zero and this means that $dV / dx = 0$. Thus, the equilibrium points are when the slope of the potential energy function is zero. Therefore, from figure 1 it is obvious that the three fixed points are x = -1 which is stable, x = +1 (stable) and x = 0 (unstable). A stable point is when the particle is given a small displacement from equilibrium, the force acting on the particle is such that the particle is attracted back to the equilibrium point where as for an unstable fixed point, the force repels the particle away from the fixed point.

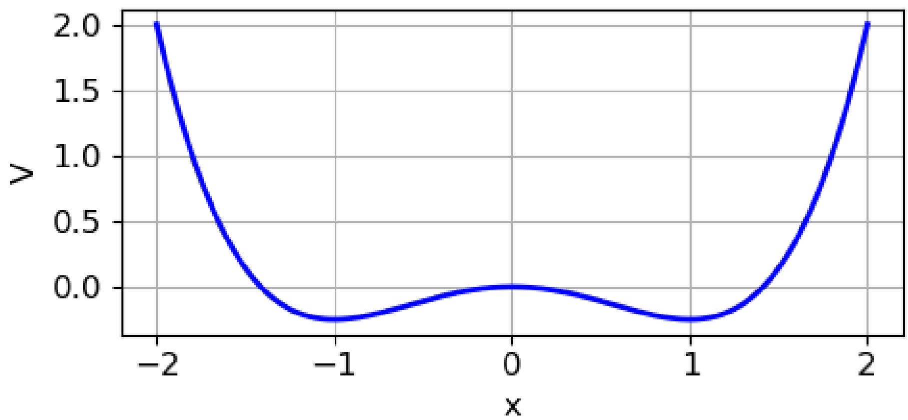


Fig. 1. Potential energy function $V(x)$ with fixed points at x = -1 (stable), x = 0 (unstable) and x = +1 (stable).

The equation of motion for the particle in the double well potential is

$$m\ddot{x} + dV / dx = 0$$

We can integrate the equation of motion with respect to time

$$m\ddot{x}\dot{x} + \left(dV / dx\right)\dot{x} = 0$$

$$\tfrac{1}{2}m\dot{x}^2 + V = \text{constant} \qquad \dot{x} \equiv v \qquad \tfrac{1}{2}m\dot{x}^2 = \tfrac{1}{2}mv^2$$

This equation states that the sum of the kinetic energy K plus the potential energy V is a constant and independent of time. The sum of the kinetic energy plus the potential energy is called the total energy E of the system.

$$E = K + V = \text{constant}$$

We say that the total energy is a conserved quantity and the system is a conservative system.

Figure 2 shows the phase portrait as a streamplot. A trajectory follows a streamline and at any point the direction of motion is tangent to the streamline. From the streamplot one can predict the motion of the particle given any initial condition. The particle may be bound to orbit around one or the other stable fixed points located at $x_e = \pm 1$.

Otherwise the orbit encompasses all three fixed points. Notice that the flow near the Origin $x_e = 0$ is always repelled.

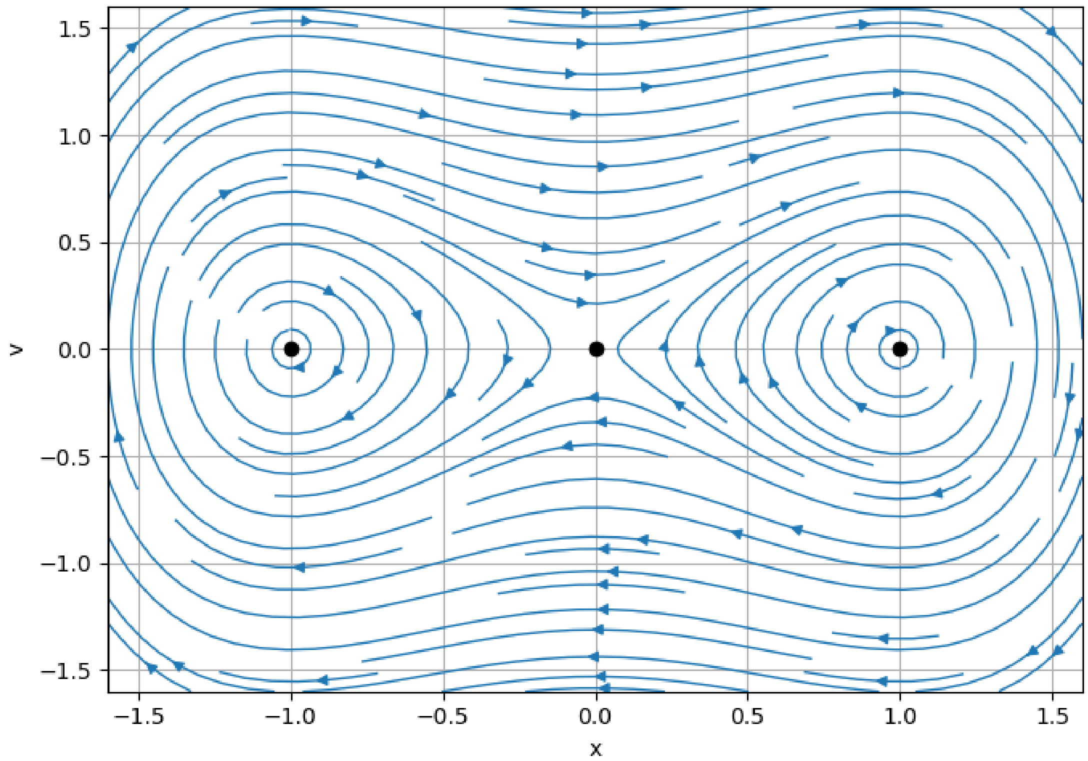


Fig. 2. Phase portrait as a streamplot. A conservative system can not have an attracting fixed point, that is, a trajectory never converges to the fixed point. The fixed points (-1, 0) and (+1, 0) are centres, and the fixed point at the Origin (0, 0) is a saddle.

Figure 3 shows the phase portrait plot of figure 2 but seven trajectories with different initial conditions are also shown. A summary of the initial conditions and the total energy for each trajectory is displayed in the Console Window:

```
 x0    y0     E
1.10  1.10  0.37
1.20  0.00  -0.20
1.20  0.70  0.04
1.20  0.60  -0.02
-1.00  -0.65  -0.04
-1.00  0.10  -0.24
-1.80  0.10  1.01
```

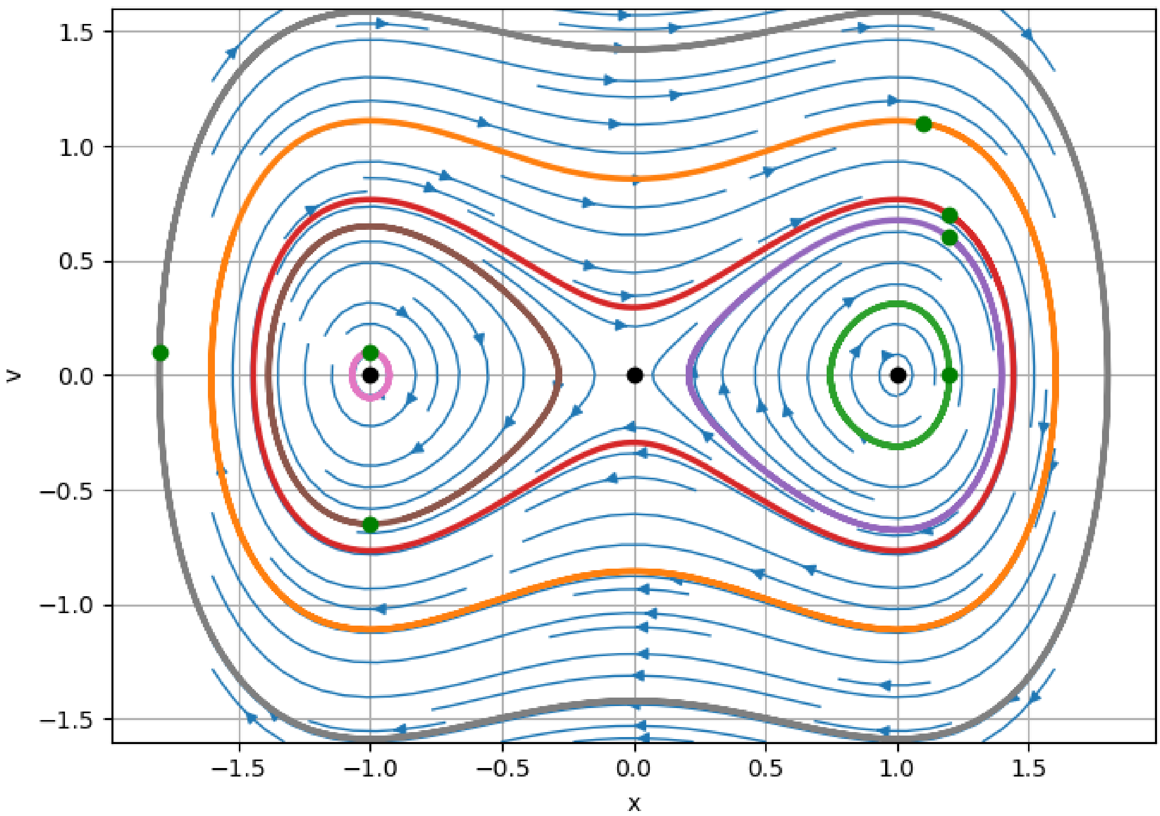


Fig. 3. Phase portrait and trajectories with different initial conditions. Green dots shown initial positions $x(0)$, $y(0)$.

For all initial conditions which are not at the fixed point, it is the total energy of the system that uniquely the phase space trajectory. As a result, trajectories can not cross each other nor merge to a fixed point. If the total energy is negative then the particle can only orbit one of the fixed points $x_e = \pm 1$. The more negative the total energy then the more strongly bound is the particle ($x(0) = -1$, $y(0) = 0.01$, $E = -0.24$ is the very tight orbit shown in magenta). For a trajectory in phase space, both the kinetic energy and the potential energy change with time but not the total energy. The total energy is a conserved quantity as shown in the time evolution plots shown in figure 4.

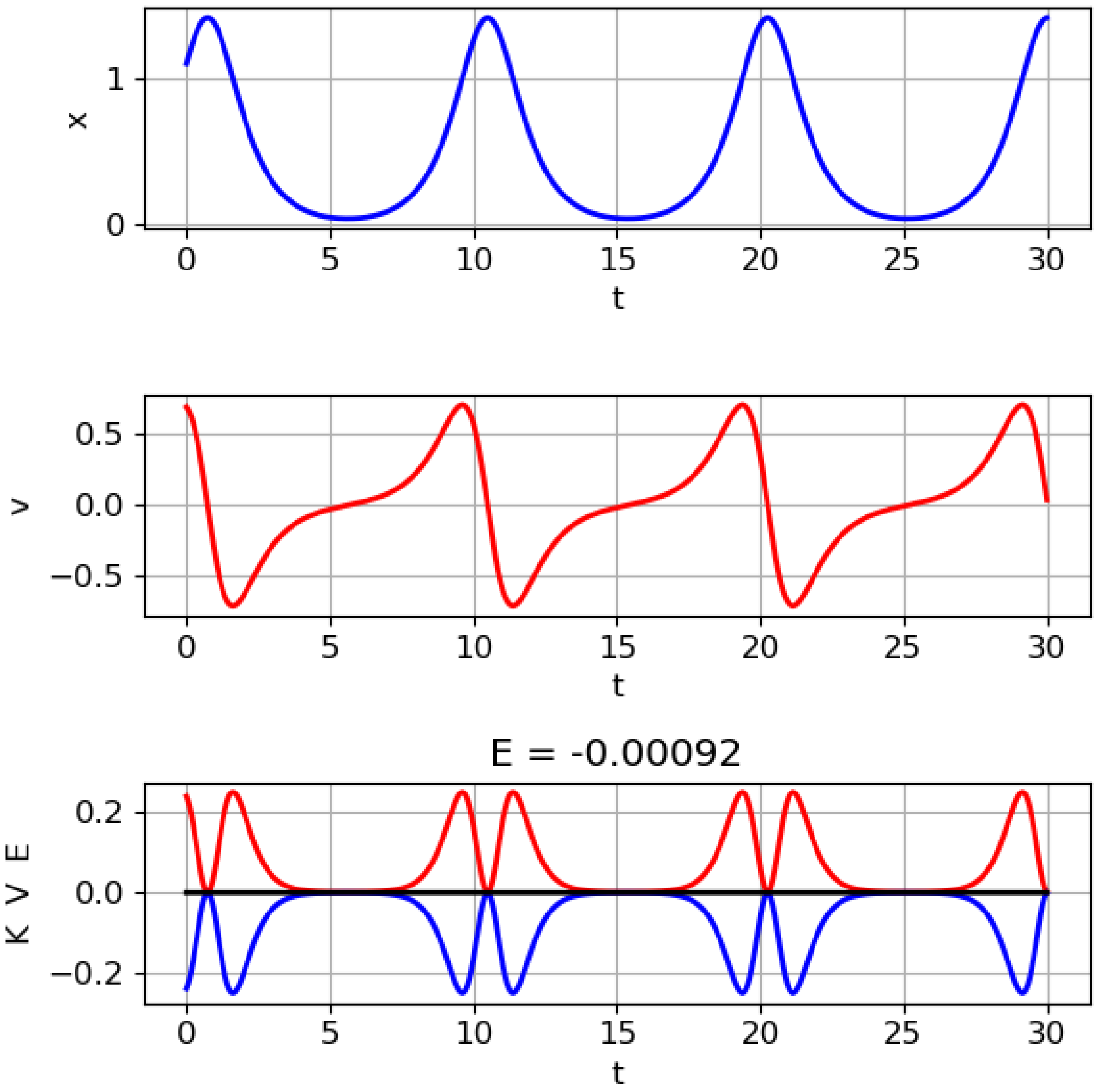


Fig. 4. Time evolution plots for displacement x, velocity v (y), kinetic energy $\boldsymbol{K}$, potential energy $\boldsymbol{V}$ and total energy $\boldsymbol{E}$.

There is an energy barrier at $x = 0$ which separates the two fixed points at $x = -1$ and $x = +1$. In figure 4, the particle is released at $x(0) = 1.1$ with velocity $v = 0.69$. The total energy of the particle is not enough to jump the barrier at $x = 0$. So, the particle simply oscillates periodically around the fixed point $x_e = +1$. As the particle approaches the barrier it gains kinetic energy at the expense of the potential energy of the system with its total energy constant. From the plots you see that the particle spends more time near the barrier than at other locations. If the particle initially is given a bigger push its total energy is increased and so can penetrate the barrier. For initial conditions *x*(0) = 1.1 and *v*(0) = 0.70, the total energy is positive (figure 5). The particle now periodically orbits all three fixed points between the limits *x* = -1.1 to *x* = 1.1.

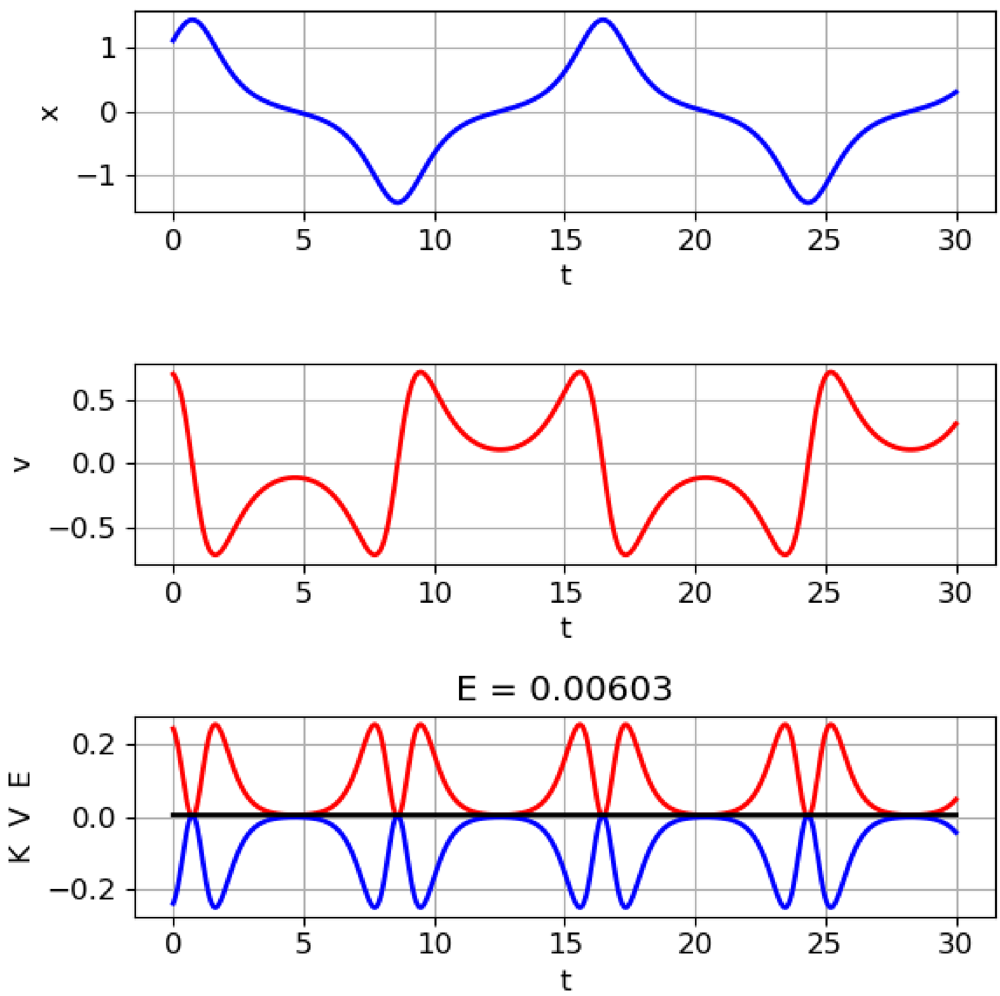


Fig. 5. Time evolution plots. The total energy is positive and can penetrate the barrier, thus periodically oscillating between -1.1 and 1.1.

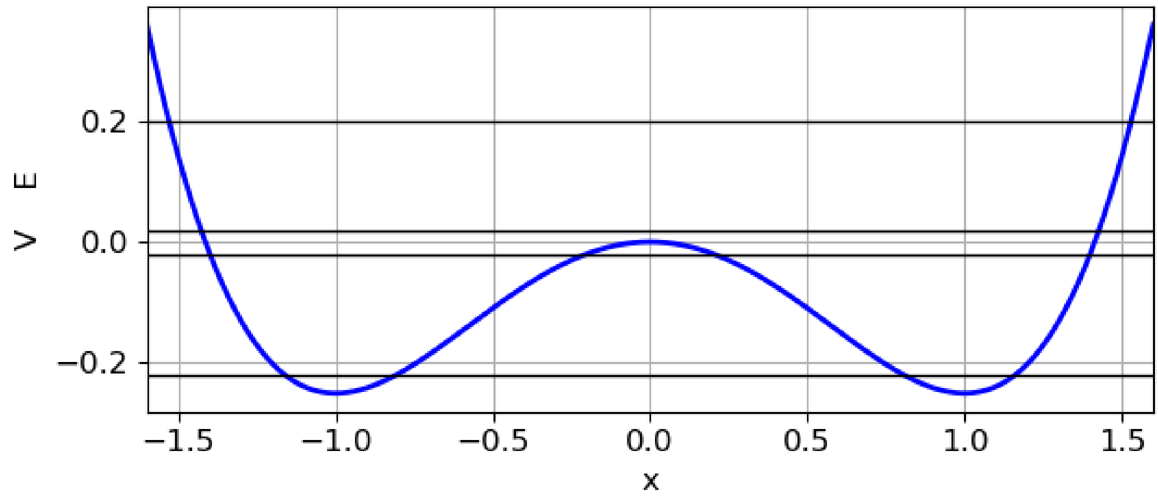


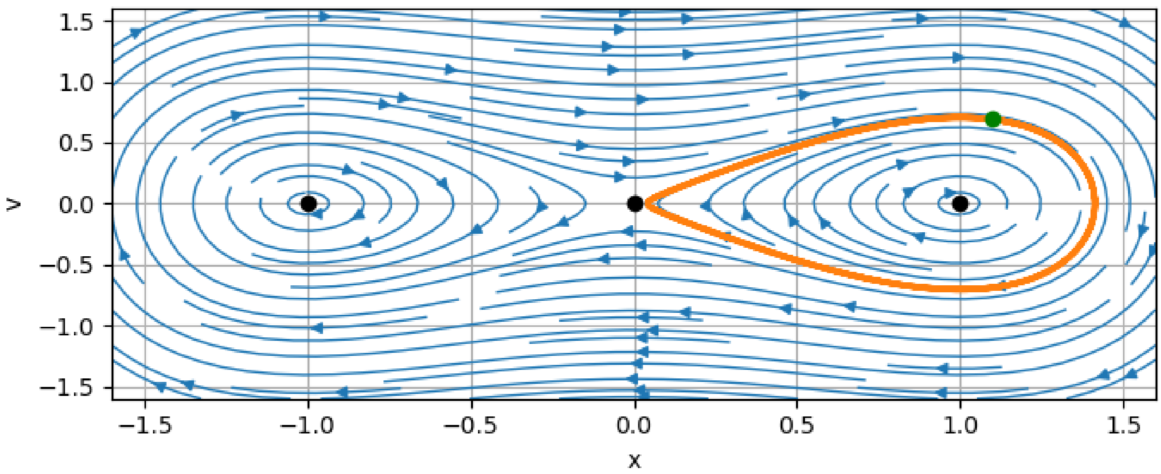
Fig. 6. Potential energy function and total energy levels for two bound states and two unbound states. The total energy is not a function of x, this total energy is the value for a trajectory specified by its initial condition $x(0)$, $y(0)$. The system becomes unbounded when the total energy is greater than the height of the barrier of the potential energy at $x = 0$.

Since $E(x,y)$ is conserved, the trajectories lie on the [1D] contours of the energy

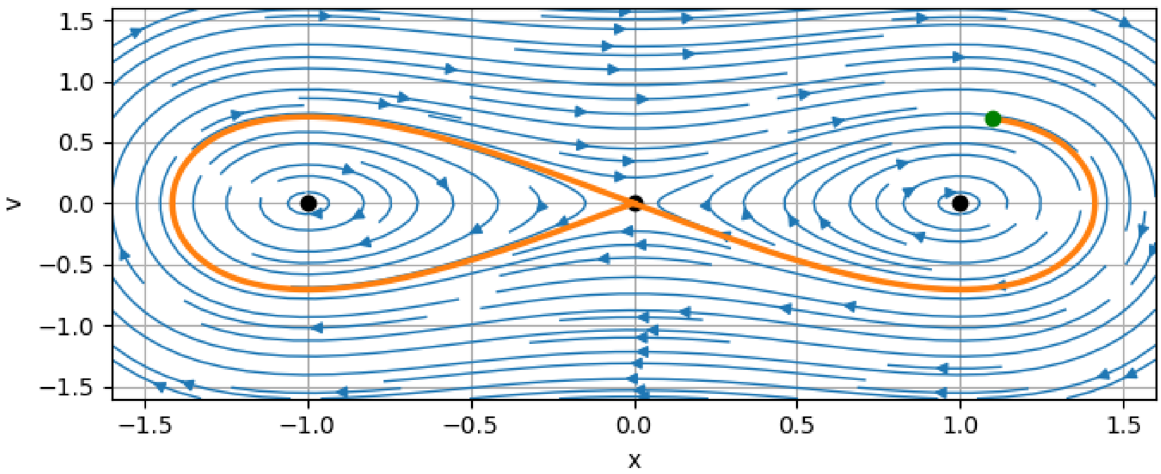
$$E(x,y) = \tfrac{1}{2}y^2 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 = \text{contant}$$

Different values of the constant determine the contour (trajectory).

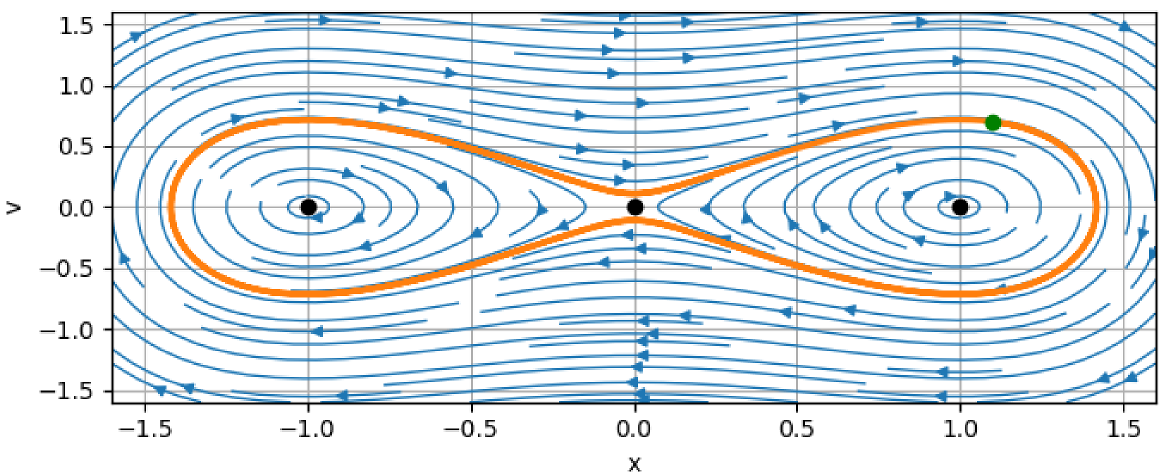
We can consider the cases where the total energy $E(x,y) \approx 0$ (figure 7).



$x(0) = 1.1 \quad y(0) = 0.69 \quad E = \text{-}0.000925$



$x(0) = 1.1 \quad y(0) = 0.6913392799487094 \quad E = 0$



$x(0) = 1.1 \quad y(0) = 0.70 \quad E = 0.006025$

Fig. 7. Saddle structure flow pattern around the fixed point (0, 0).

The Origin (0, 0) is a **saddle**. Examination of the flow pattern near the Origin and you see flow lines going in and flow lines going out which is a typical saddle structure. The orbits are called homoclinic orbits. The are repelled from the saddle back are then drawn back to it.

REFERENCES

Jason Bramburger

Conservative Systems - Dynamical Systems | Lecture 18

https://www.youtube.com/watch?v=gKYSxW_syt0

Dynamics of the Double-Well Duffing System

https://www.itm-conferences.org/articles/itmconf/pdf/2025/06/itmconf_iconmaa25_02002.pdf