DOING PHYSICS WITH PYTHON

DYNAMICAL SYSTEMS

DAMPED SIMPLE PENDULUM

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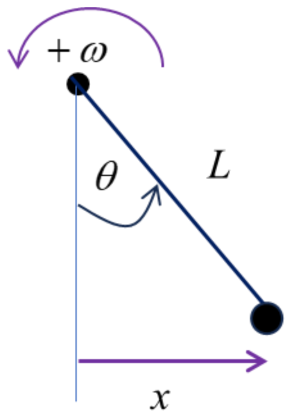
ds1900.py

A damped simple pendulum is a real-world pendulum that eventually stops swinging due to energy loss from forces like air resistance and friction. While an ideal, frictionless pendulum would swing forever, the damping in a real pendulum causes the amplitude of its swing to gradually decrease over time until it comes to rest.

We will consider the simple rigid pendulum of length L that is constrained to move along an arc of a circle centred at a pivot point. The angular displacement w.r.t. the vertical is θ, the angular velocity is ω, and the horizontal displacement is x. The mass of the pendulum rod is taken as massless and the mass m of the system is concentrated at the bob of the pendulum.



The equation of motion for the simple damped pendulum is

$$(1)\qquad \begin{aligned} m\frac{d^2\theta}{dt^2} &= -\left(\frac{m\,g}{L}\right)\sin(\theta) - b\,m\frac{d\theta}{dt} \\ \frac{d^2\theta}{dt^2} &= -\left(\frac{g}{L}\right)\sin(\theta) - b\,\frac{d\theta}{dt} \end{aligned}$$

where g is the acceleration due to gravity, and b is the damping coefficient.

To solve equation 1 using the Python function **odeint**, we need to write this second-order equation as the system of two first-order equations

$$(2)\qquad \begin{aligned} \frac{d\theta}{dt} &= \omega \\ \frac{d\omega}{dt} &= -\left(\frac{g}{L}\right)\sin(\theta) - b\,\omega \end{aligned}$$

For the Python Code, the variables x and y are used where

$$\theta \to x \qquad \omega \to y \qquad g / L \to w^2$$

```
def lorenz(t, state):
   x, y = state
   dx = y
   dy = -b*y - w**2*sin(x)
   return [dx, dy]
```

For small amplitude free vibrations of the simple pendulum (zero damping), its natural period T_0 and frequency f_0 of vibration are

$$(3) \qquad T_0 = 2\pi\sqrt{\frac{L}{g}} \qquad f_0 = \left(\frac{1}{2\pi}\right)\sqrt{\frac{g}{L}} \qquad \omega_0 = 2\pi\, f_0 = \frac{2\pi}{T_0}$$

The two fixed-point of the system occur when $d\omega / dt = 0$ and $d\theta / dt = 0$. Hence, the two fix-points of the system are

$\omega = 0 \qquad \theta = 0^{\circ} \quad (0, -L)$ stable equilibrium point

$\omega = 0 \qquad \theta = 180^{\circ} = \pi$ rad $\;(0, L)$ unstable equilibrium point

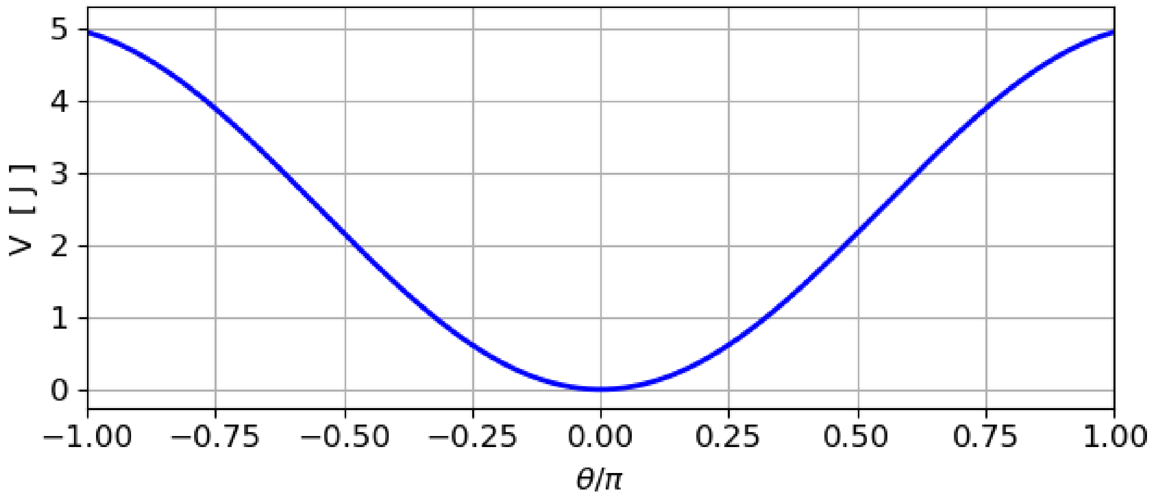


Fig. 1. Potential energy function $V(\theta)$ with fixed points at $\theta = \pm\pi$ (**unstable**), and $\theta = 0$ (**stable**).

Free motion of pendulum (zero damping b = 0)

For zero damping, there is no dissipation of energy. Therefore, it is a conservative system and the total energy is constant. The constant E_0 is only determined by the initial conditions $(\theta(0), \omega(0))$. The sum of the kinetic energy K plus the potential energy V is called the total energy E of the system.

$$E_0 = K + V = \text{constant}$$

Figure 2 shows the phase portrait as a streamplot. A trajectory follows a streamline and at any point the direction of motion is tangent to the streamline. From the streamplot one can predict the motion of the particle given any initial condition.

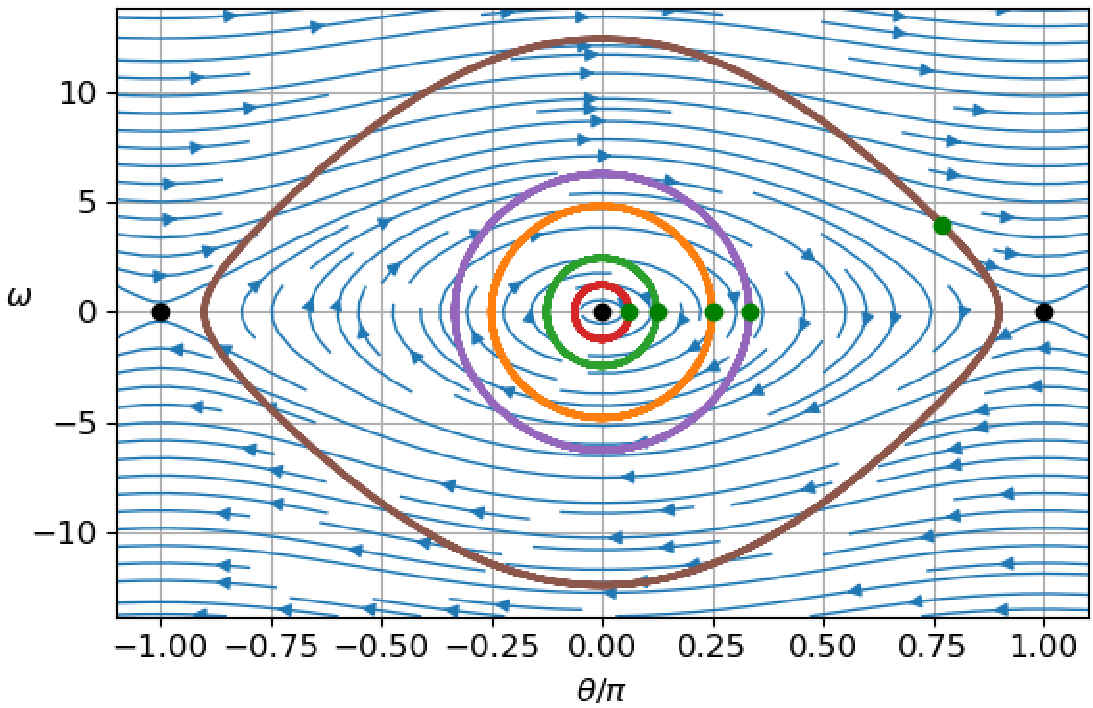


Fig. 2. Phase portrait as a streamplot. Five trajectories with different initial conditions are shown. Note: a trajectory is uniquely determined by its initial conditions and hence initial total energy. Hence, trajectories can never cross. For the trajectories shown, the pendulum simply swings back and forth for ever. Green dots shown the initial conditions $(\theta(0), \omega(0))$.

We can solve the system ODE to plot the time evolution for the angular displacement θ, angular velocity $\dot{\theta} \equiv \omega$, and energies K, V and E (figure 3) for different initial conditions $(\theta(0), \omega(0))$. Figure 3 shows a series of plots for $\theta(0) = \pi / 8$ and $\omega(0)$ is used as a control parameter.

The default parameters were chosen so the natural frequency of oscillation is 1.00 s

$$g = 10.0 \quad T_0 = 1.0 \quad \omega_0 = \frac{2\pi}{T_0} \quad L = \frac{g}{{\omega_0}^2} = 0.2533$$

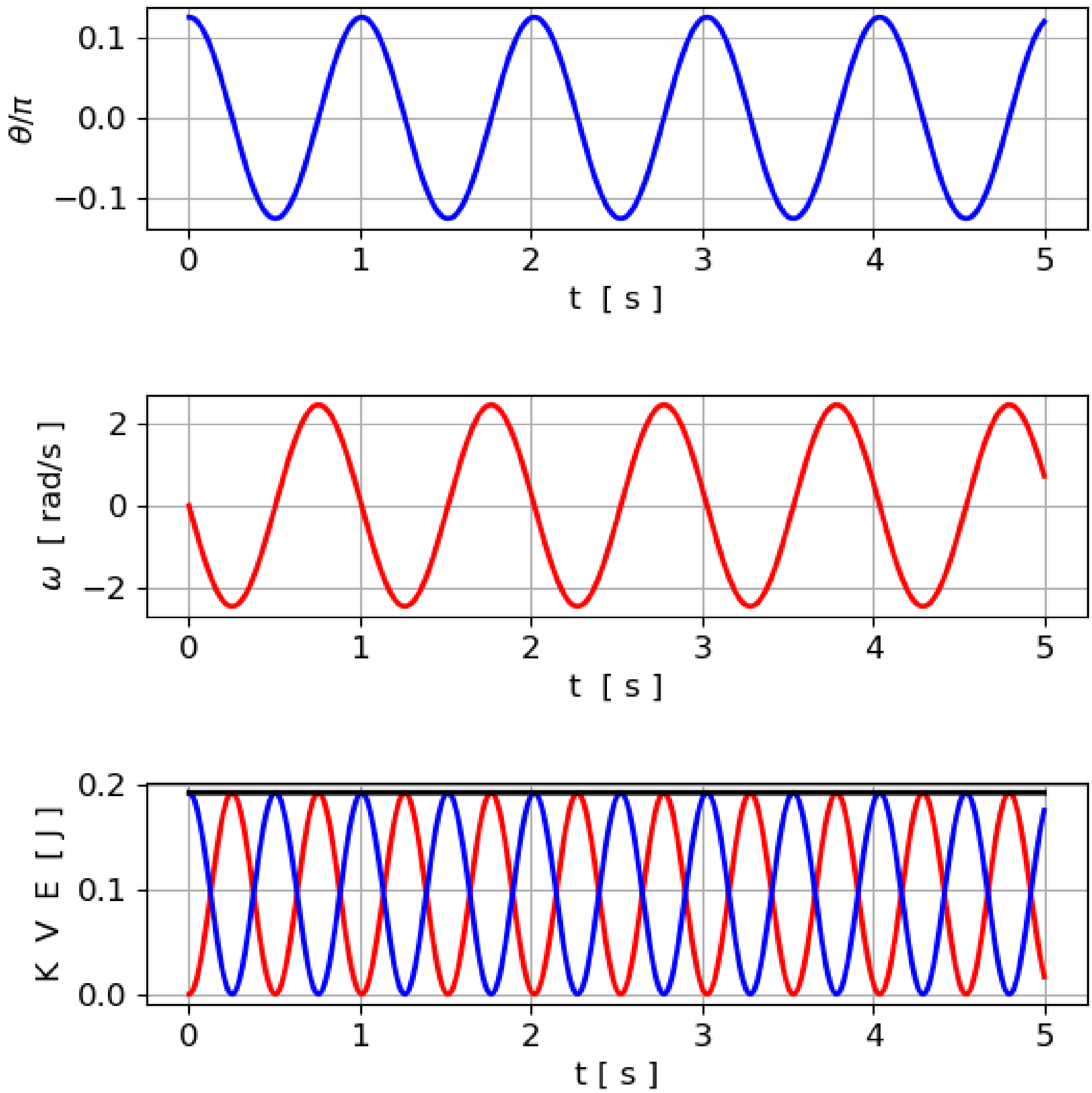


Fig. 3A. $\left(\theta(0) = \pi / 8, \omega(0) = 0\right)$ Small amplitude oscillations. The motion is simple harmonic motion and the system oscillates at its natural frequency with period T_0 = 1.00 s and total energy E = 0.193. Energy graph: ***K*** (red), ***V*** (blue), ***E*** (black)

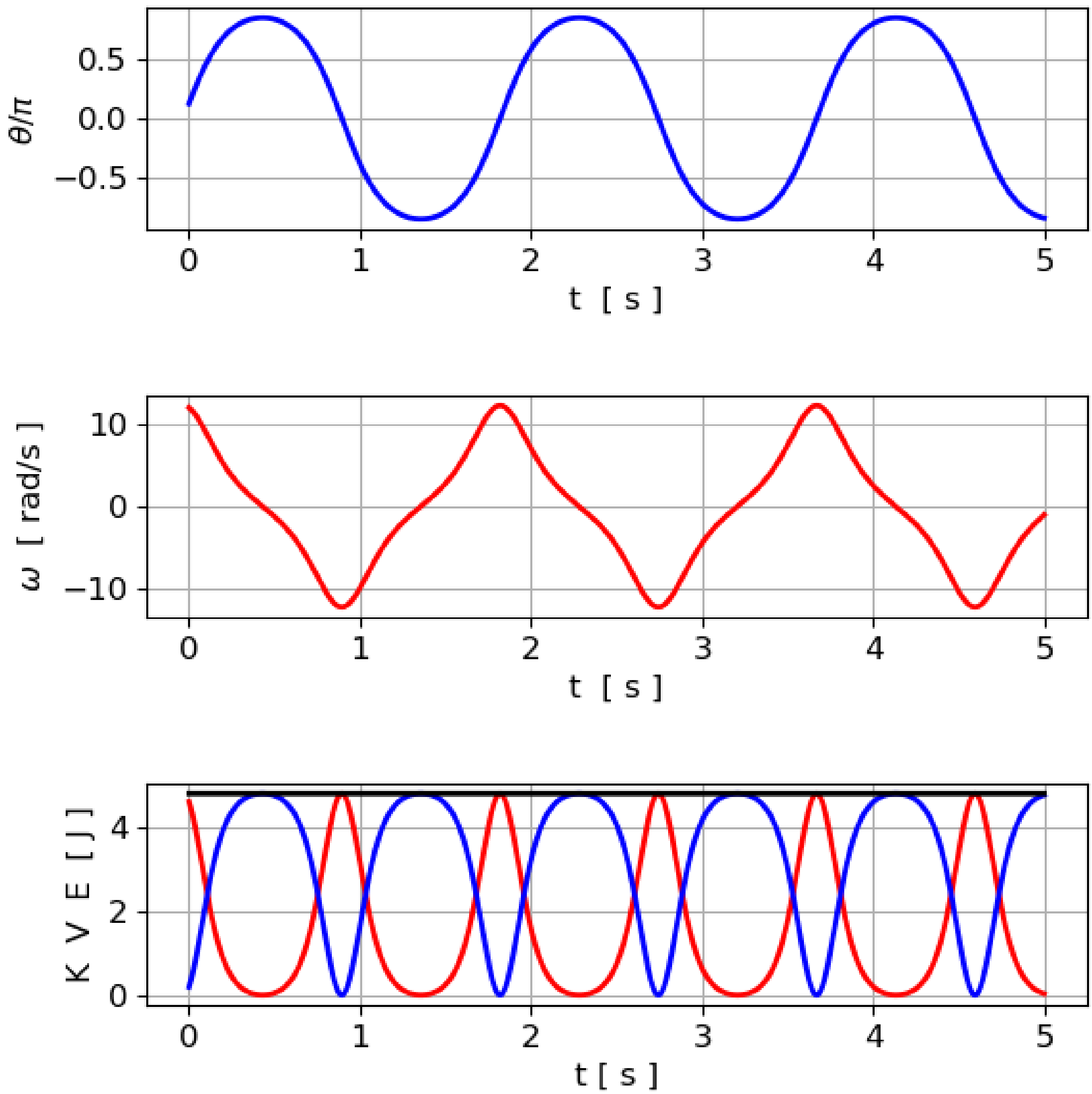


Fig. 3B. $\left(\theta(0) = \pi / 8, \omega(0) = 12\right)$ Large amplitude oscillations. The motion is not simple harmonic motion and the system does not oscillate at its natural frequency with period $T_0 = 1.00$ s. The total energy $E = 4.813$.

We can give the pendulum a greater push so that it can just reach the vertical position $\theta = \pm\pi$ where the potential energy is $2mgL$ and kinetic energy is zero

$$2\,m\,g\,L = \tfrac{1}{2}\,m\,(L\,\omega(0))^2 + m\,g\,L\,(1-\cos(\pi/8))$$

So, the required initial angular velocity is

$$\omega(0) = 12.3249$$

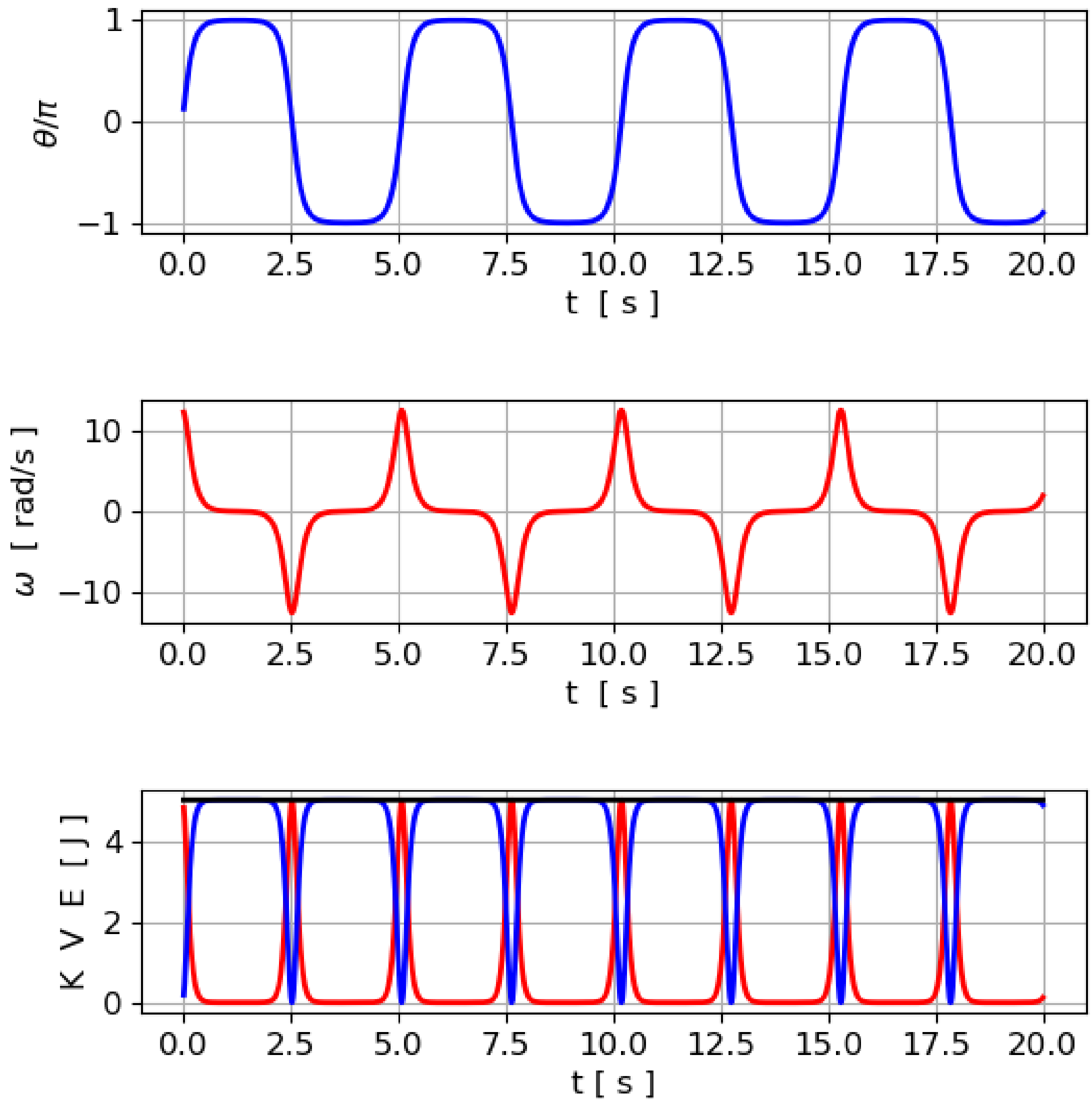


Fig. 3C. $\left(\theta(0) = \pi/8, \omega(0) = 12.3249\right)$ Large amplitude oscillations. The pendulum almost reaches the vertical position before it falls back and swings to the bottom of the arc before rising again to almost vertical. The pendulum is swinging very slowly at the top of its motion. The total energy $E = 5.066$.

When we increase the initial kinetic energy again, the pendulum no longer swings back and forth but simply keeps rotating about the pivot point.

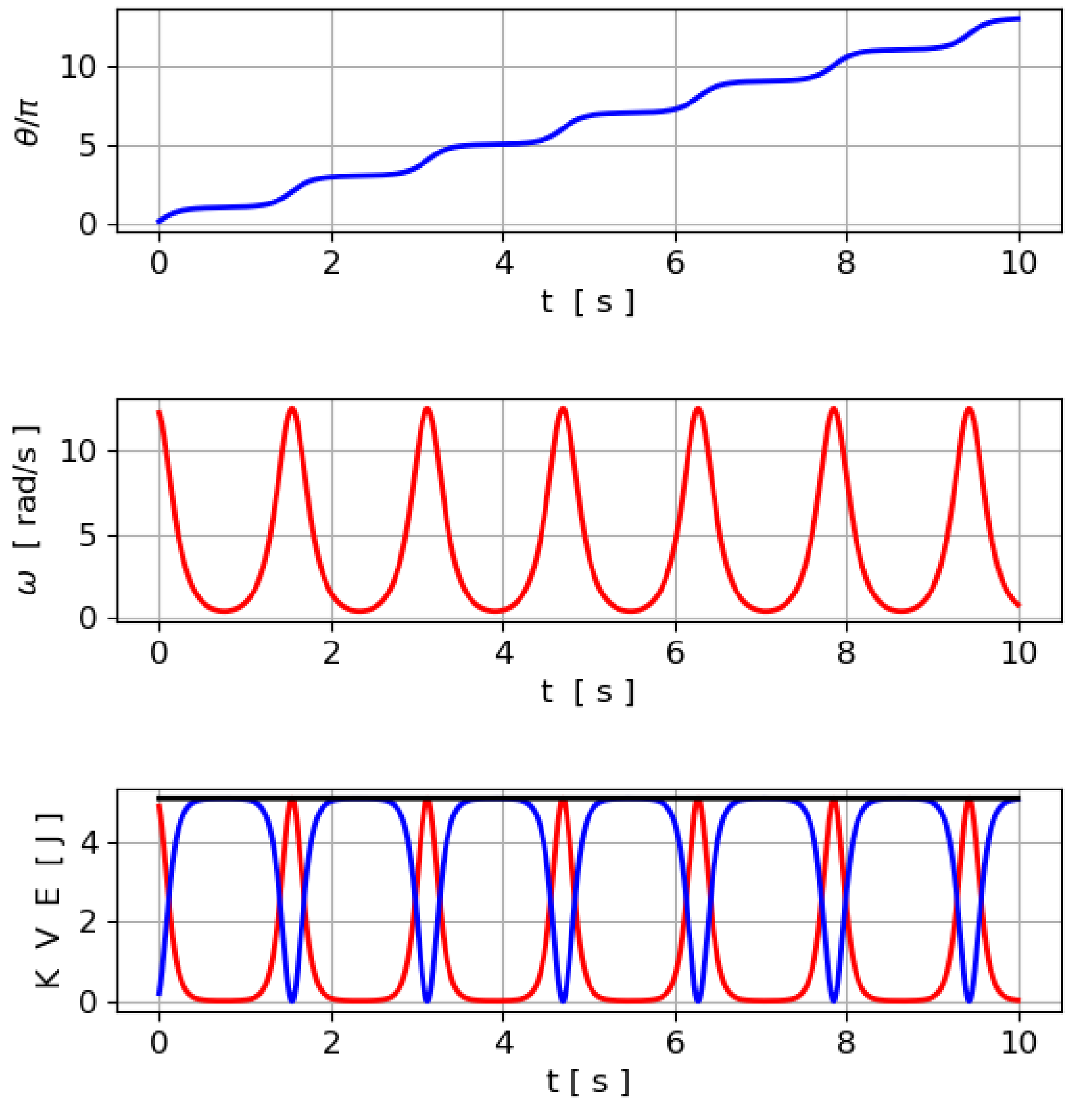


Fig. 3D. $\left(\theta(0)=\pi/8, \omega(0)=12.33\right)$ Large amplitude oscillations. The pendulum now passes the vertical and simply keeps rotating around the pivot point. The total energy $E = 5.070$.

The initial angular velocity is a bifurcation parameter and the bifurcation point is $\omega_B = \omega(0) = 12.324911326655178$.

$$\omega(0) < \omega_B \;\Rightarrow\; \text{pendulum swings back to forth}$$

$$\omega(0) > \omega_B \;\Rightarrow\; \text{pendulum rotates}$$

Damped motion of pendulum (zero damping b = 0.80)

For the damped motion of the pendulum, system loses energy, typically converting it into thermal energy, causing the oscillations to decay. Energy is not conserved and we have a non-conservation where the total energy decreases to zero.

Figure 4 shows the plots of the time evolution plots for the damped pendulum system and figure 5 shows the vector field and the phase space trajectories for three different initial conditions.

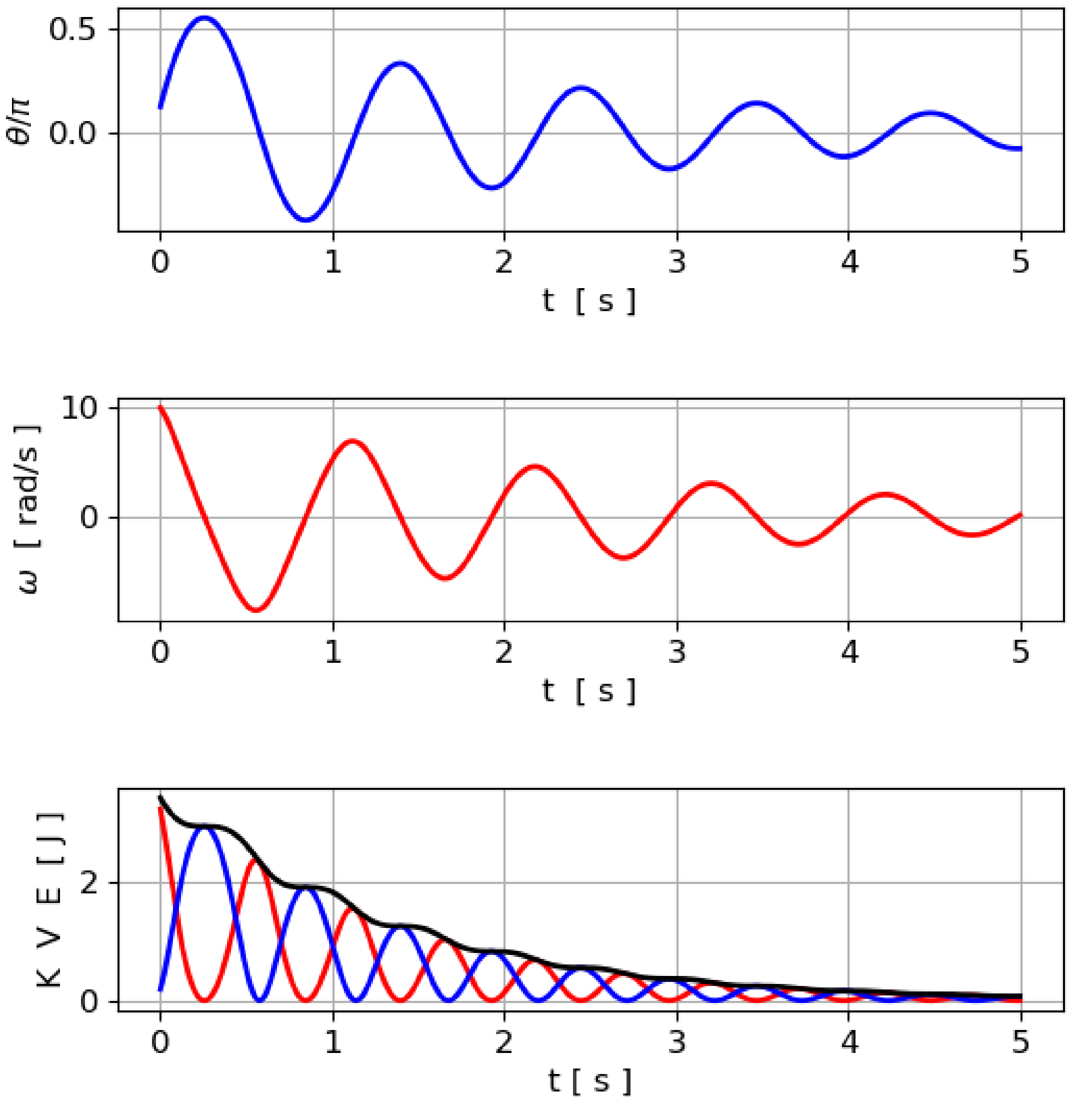


Fig. 4. Time evolution plots for the damped pendulum system. Energy is dissipated and the oscillations die away and eventually the pendulum will come to rest at $\theta = 0$, the stable equilibrium point of the system.

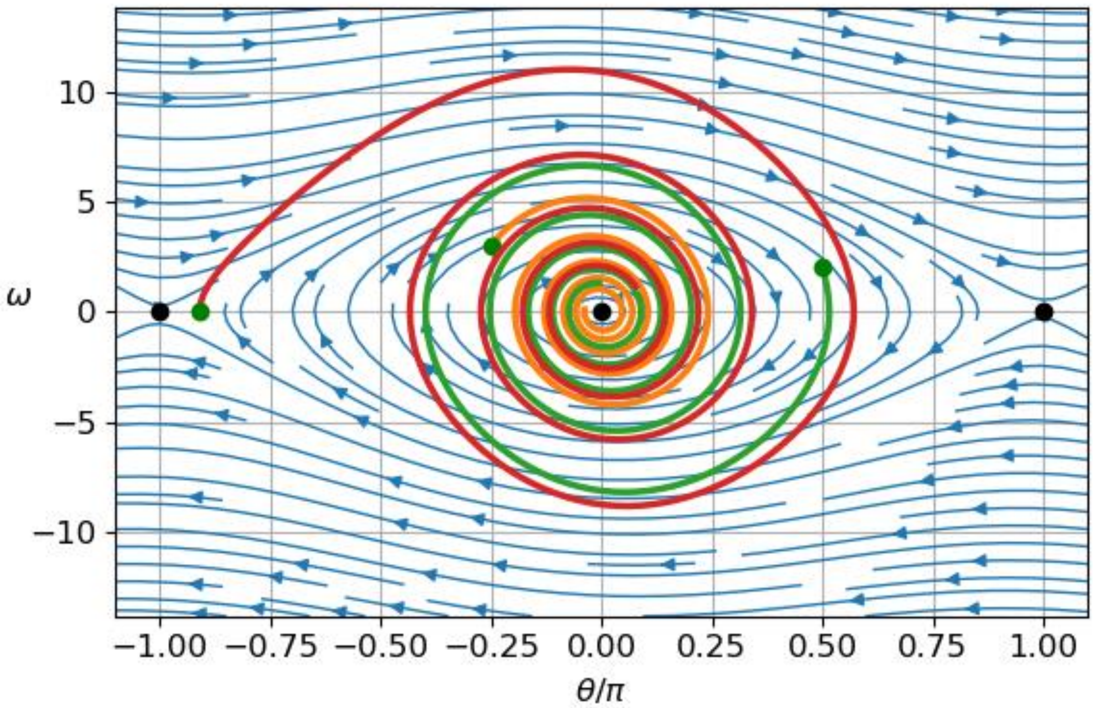


Fig. 5. Vector field for the damped pendulum system and three trajectories in phase space. All three trajectories converge to the fixed point (0, 0).

REFERENCES

Jason Bramburger

The Pendulum - Dynamical Systems | Lecture 19

https://www.youtube.com/watch?v=9Ler7LvnIoA&t=1111s