DOING PHYSICS WITH PYTHON

DYNAMICAL SYSTEMS

VAN DER POL OSCILLATOR

Limit cycles

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ds2100.py

LIMIT CYCLE

A **limit cycle** is a closed trajectory in a nonlinear dynamical system's phase space where nearby trajectories either spiral into it (stable limit cycle) or spiral out of it (unstable limit cycle). Unlike linear oscillations, the amplitude and frequency of a limit cycle are constant and independent of initial conditions, and it is a characteristic behaviour of nonlinear systems. A common example is the Van der Pol oscillator, which was discovered in electrical circuits and is used to model phenomena like nerve cell action potentials

Examples and applications:

- Electrical circuits
- Predator-prey models
- Modelling the periodic firing of neurons (action potentials) and other biological rhythms.
- The periodic motion of mechanical devices (clock's escapement mechanism) which rely on limit cycles to maintain a constant amplitude of oscillation.

Van der Pol Oscillator

The Van der Pol oscillator is a self-sustained, nonlinear oscillator known for its nonlinear damping and it oscillates without an external force due to its nonlinear damping. At small amplitudes, it gains energy (like a negative damping), while at large amplitudes, it dissipates energy. The oscillator's trajectory in phase space eventually converges to a closed loop, which represents its stable, periodic oscillation. Its behaviour can be described by a second-order differential equation and it exhibits a limit cycle, meaning it will eventually settle into a stable oscillation. The second-order non-linear autonomous differential equation

$$(1) \qquad \ddot{x} = -x + \mu\left(1 - x^2\right)\dot{x} + A\cos\left(2\pi t / T_{IN}\right) \qquad \mu \geq 0$$

$$\ddot{x} = \underbrace{-x}_{\text{restoring term}} + \underbrace{\mu\left(1-x^2\right)\dot{x}}_{\substack{\text{damping term} \\ \mu \geq 0}} + \underbrace{A\cos\left(2\pi t / T_{IN}\right)}_{\text{sinusoidal forcing term}}$$

is called the **Van der Pol equation**. This form of the van der Pol equation includes a periodic forcing term and results in deterministic chaotic motion. The Van der Pol equation describes many physical systems. The parameter μ is a positive scalar indicating the nonlinearity and the strength of the damping. The sign of the damping term in equation 1 is dependent upon the sign of the term $\left(1-x^2\right)$. The equation models a non-conservative system in which energy is added to $\left(|x|<1\right)$ and subtracted from $\left(|x|>1\right)$ the system, resulting in a periodic motion called a **limit cycle**. Hence, energy is dissipated at high amplitudes and generated at low amplitudes. As a result, there exists oscillations around a state at which energy generation and dissipation balance.

To solve the Van der Pol equation using the Python function **odeint**, equation 1 needs to be expressed as two first order differential equation.

$$\text{(2)} \qquad \begin{aligned} \dot{x} &= y \\ \ddot{y} &= -x + \mu\left(1 - x^2\right)y \end{aligned}$$

The results of solving equation 2 show that for every initial condition (except $x = 0$, $\dot{x} = 0$), approaches a unique periodic motion. The nature of this limit cycle is dependent on the value of μ. The limit cycle is a closed curve enclosing the Origin in the *x-y* phase plane. The limit cycle is also symmetrical about the Origin. When $\mu = 0$, $A = 0$ the motion is simple harmonic motion. For small values of μ the motion is nearly sinusoidal, whereas for large values of μ it is a relaxation oscillation, meaning that it tends to resemble a series of step functions, jumping between positive and negative values twice per cycle.

SIMULATIONS

The variable x could represent many different physical quantities such as voltage, current, displacement and the y variable is the time rate of change of x. For the simulations, to make it less abstract, we can take x to be the displacement and y to be the velocity v of a particle.

Free motion (simple harmonic motion) $\mu = 0$

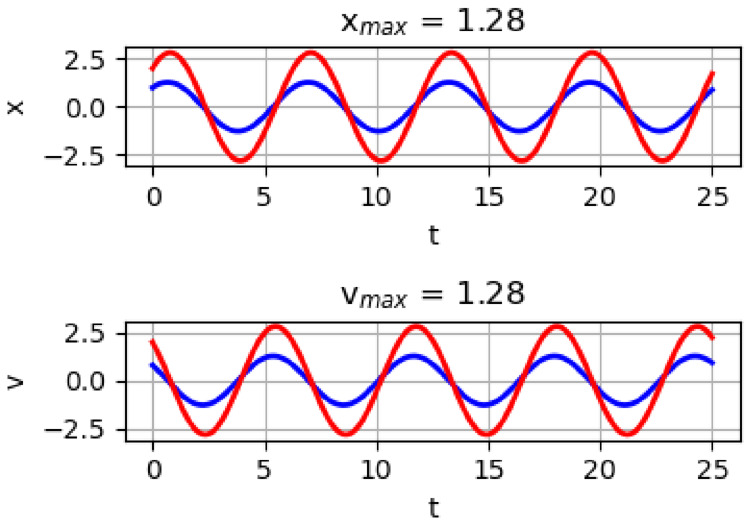

Fig. 1.1. $\mu = 0$ The motion of the particle is simple harmonic motion (SHM) for to initial conditions. The period is $T = 6.28$ and is the same for all initial conditions, excluding the Origin.

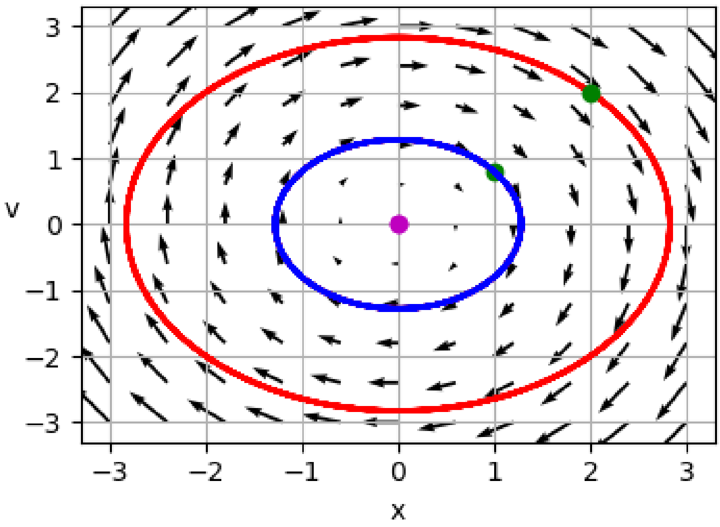

Fig. 1.2. $\mu = 0$ Phase portrait as a quiver plot showing the flow of the system. The Origin (0, 0) is an unstable fixed point. The orbital period is $T = 6.28$ and the same for all initial conditions. So, the larger orbit must have an average velocity greater the smaller orbit trajectory. The orbits are purely elliptical.

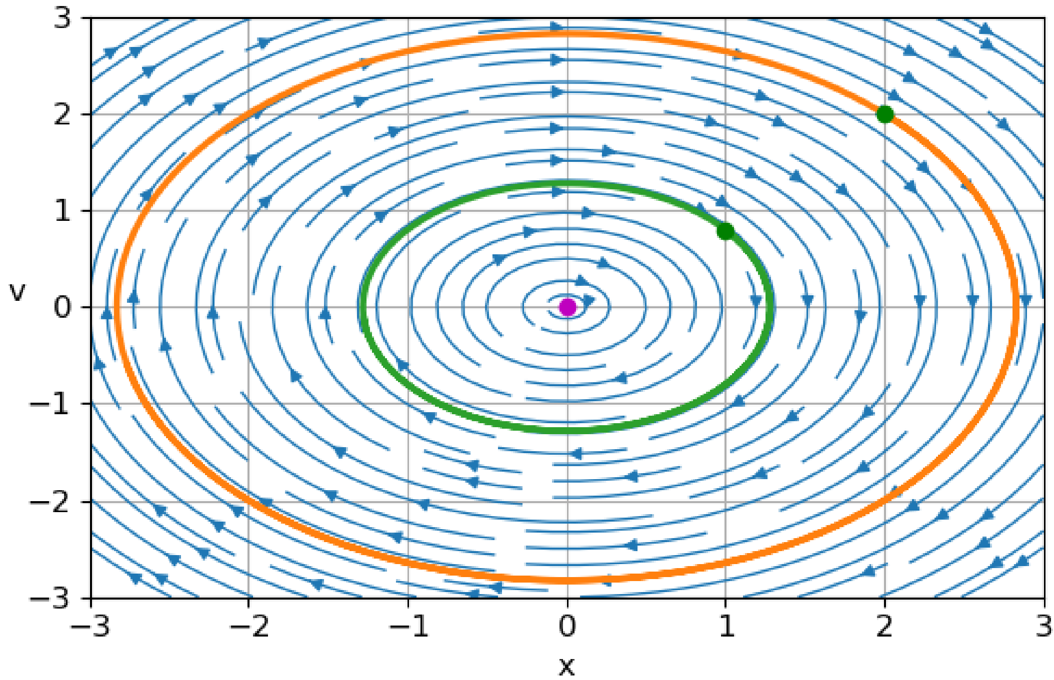

Fig. 1.3. $\mu = 0$ Phase portrait as a streamplot showing the flow of the system. The Origin (0, 0) is an unstable fixed point.

For the case when the damping coefficient is zero, $\mu = 0$, the motion is simple harmonic motion and the phase space orbital period is a constant, $T = 6.28$. The period T is independent of the initial conditions or the phase space trajectory.

Weak nonlinear damping $\mu = 0.1$

For weak nonlinear damping, the value of the damping coefficient μ is small and all trajectories are attracted towards a stable limit cycle. If the initial x_0 value is small then the orbit it is pushed out to the limit cycle and if x_0 is large then the orbit is pulled into the limit cycle. The oscillations are symmetrical about the Origin.

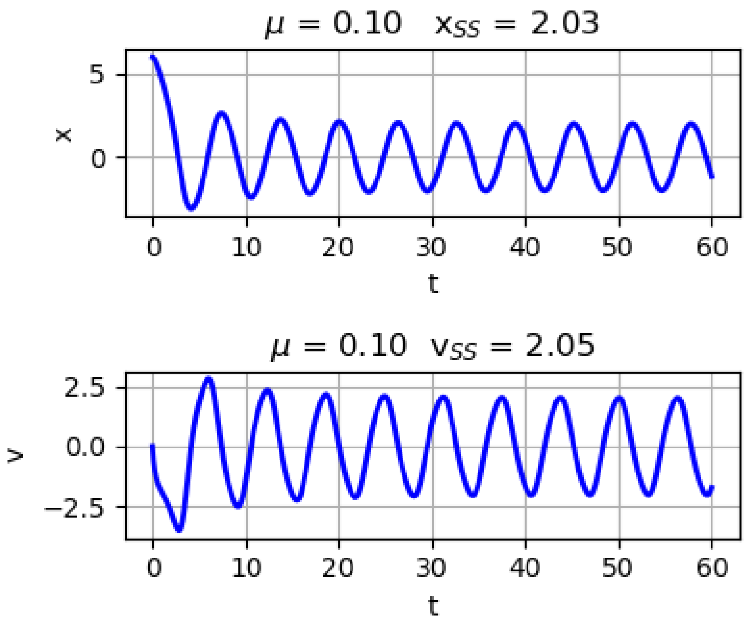

Fig. 2.1. Trajectory pulled into limit cycle. The orbital period is $T = 6.28$. The amplitude of the oscillation decreases to its steady-state amplitude x_{SS}.

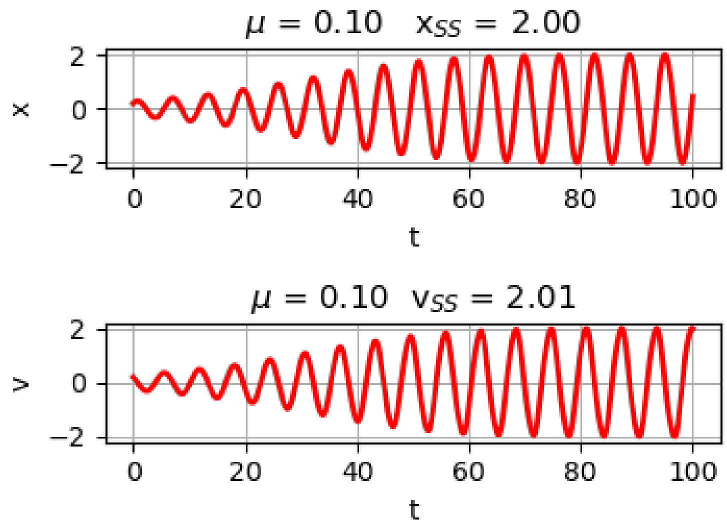

Fig. 2.2. Trajectory pushed out to limit cycle. The amplitude grows to its steady-state value x_{SS}.

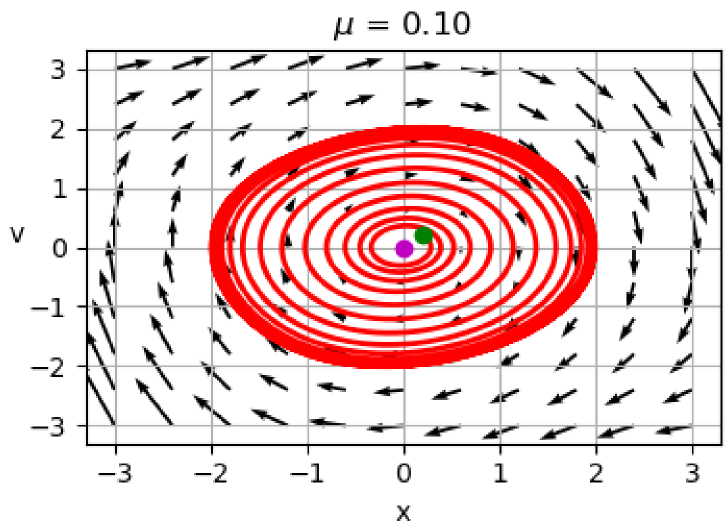

Fig. 2.3. The orbit spirals outward to the limit cycle.

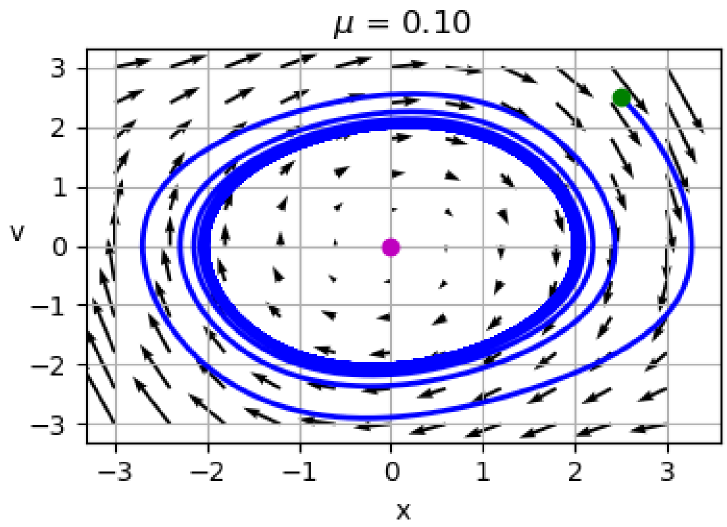

Fig. 2.4. The orbit spirals inward to the limit cycle.

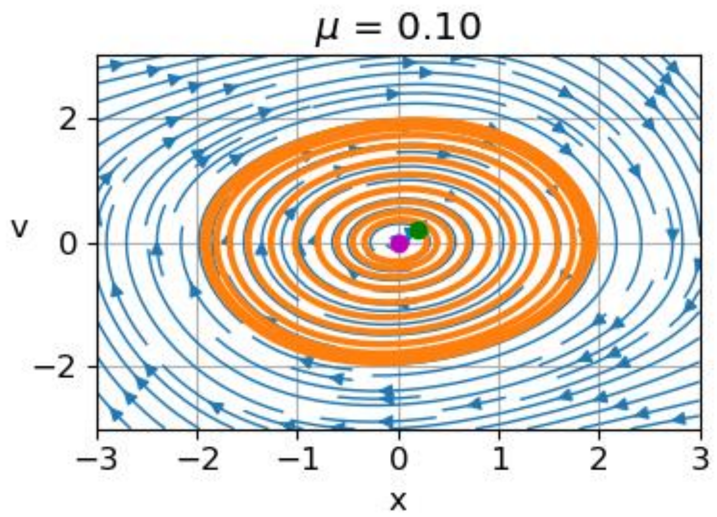

Fig. 2.5. The orbit spirals outward to the limit cycle.

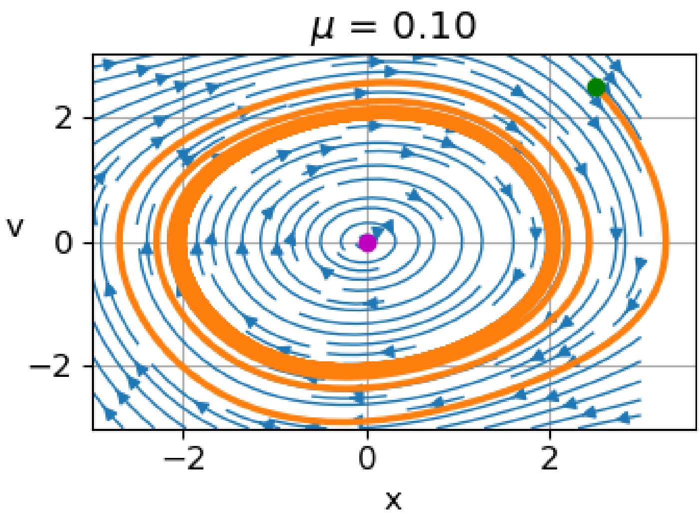

Fig. 2.6. The orbit spirals inward to the limit cycle.

For $0 < \mu < 1$ the phase space is a small distortion of the phase space of the harmonic oscillator and there is still an elliptical behaviour of the orbits.

Moderate nonlinear damping $\mu = 2.0$

For moderate nonlinear damping all trajectories are attracted towards a stable limit cycle but the shape of the decaying orbit is no longer "elliptical" in shape. Again, if the initial x_0 value is small then the orbit it is pushed out to the limit cycle and if x_0 is large then the orbit is pulled into the limit cycle. The oscillations are symmetrical about the Origin.

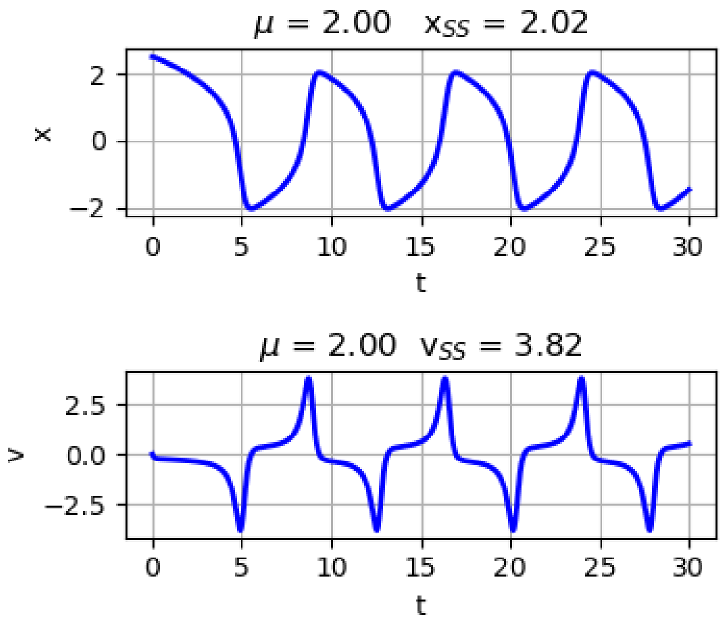

Fig. 3.1. Trajectory pulled into limit cycle. The motion is periodic but not SHM. The orbital period is greater than the free oscillations, $T = 7.63$.

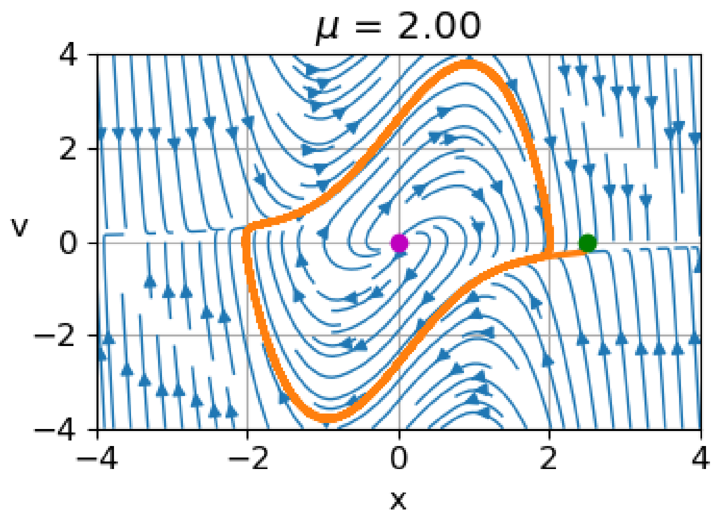

Fig. 3.2. A trajectory is attracted to the limit cycle.

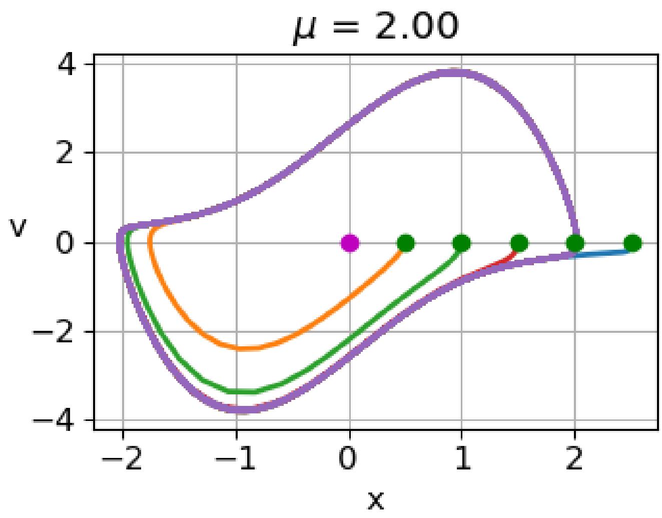

Fig. 3.3. All trajectories with different initial conditions are drawn into the limit cycle.

Strong nonlinear damping $\mu = 10.0$

For strong nonlinear damping where $\mu \gg 1$ the system behaves as a relaxation oscillator (system response is a non-sinusoidal repetitive output signal).

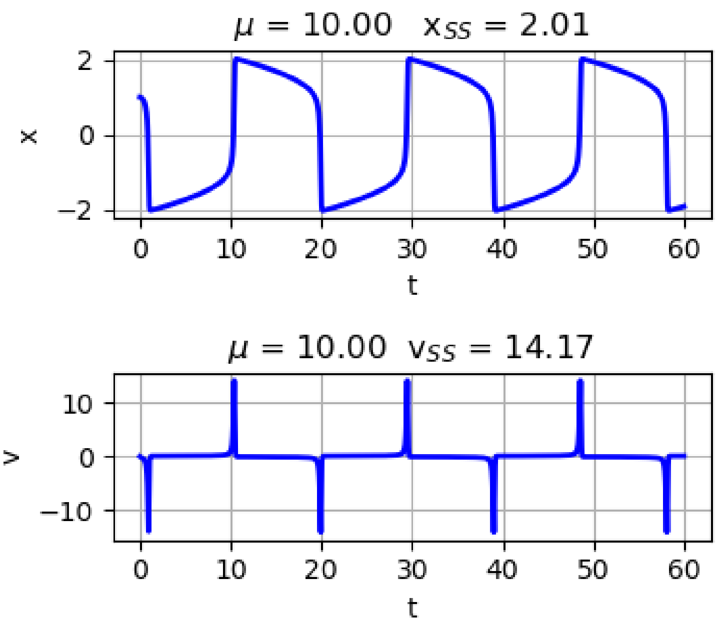

Fig. 4.1. Time evolution plots.

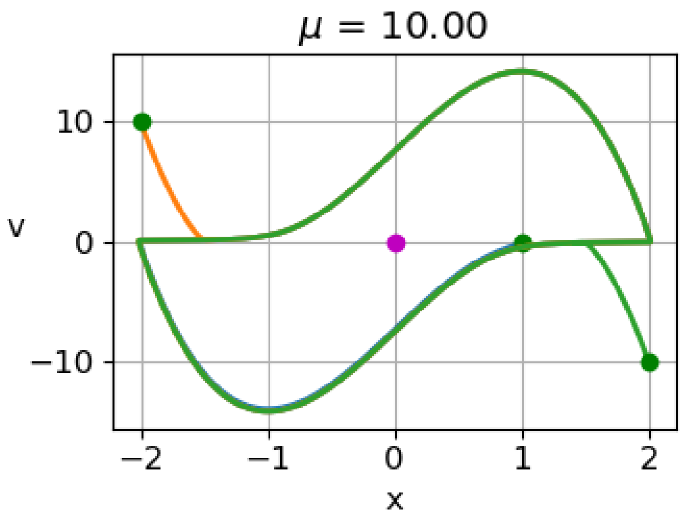

Fig. 4.2. All final trajectories merge to the limit cycle. The limit cycle for $\mu \gg 1$ is elongated and the period becomes large $T \sim 20$.

Figure 5 show a series of phase portraits for different values of the damping coefficient μ. The amplitude of the displacement x $\left(x_{SS} \approx 2\right)$ is constant for all values of μ. Increasing the value of μ increases the velocity amplitude and increases the period of oscillation.

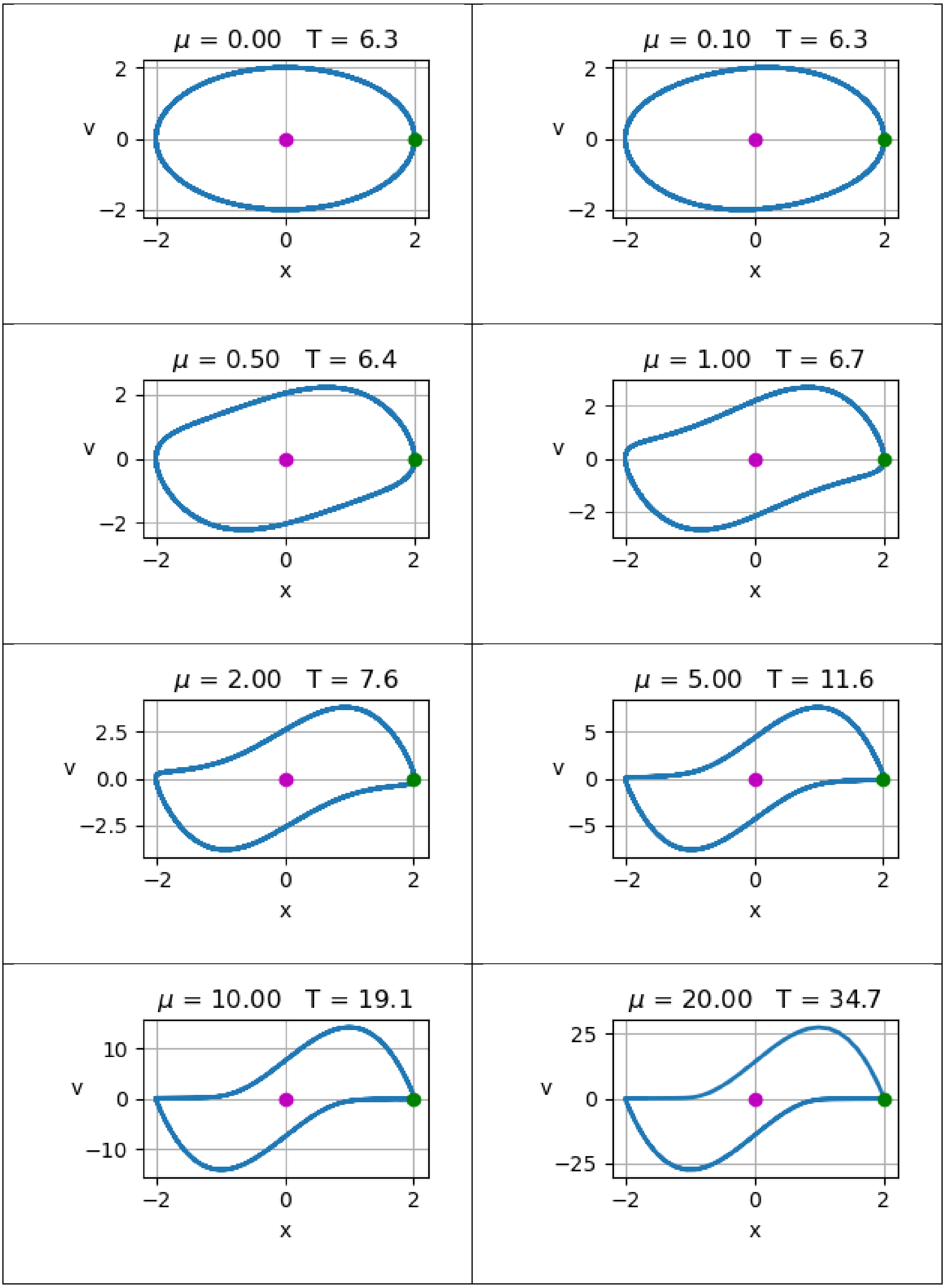

Fig. 5. Orbits for different values of the damping coefficient μ.

The motion with strong nonlinear damping is divided into two time scales

Fast time scale: rapid transitions $x \quad -2 \to +2 \quad +2 \to -2$

Slow time scale: $x \sim 2 \quad x \sim -2 \quad T > 20$ time units

A Van der Pol oscillator is one in which all initial conditions converged to the same periodic orbit of finite amplitude. This dynamical system has a unique stable limit cycle where there is a unique periodic solution and all nearby solutions tend towards this periodic solution as $t \to \infty$.

When $|x| >> 1$ both the restoring and damping forces are large, so that $|x(t)|$ will decrease with time. The system behaves like a strongly damped oscillator and it disperses energy. However, when $|x| << 1$ the damping force becomes negative, which makes $|x(t)|$ increase with time and the energy of the system grows.

The Python Code **ds2100.py** can easy be modified to include the forcing term.

REFERENCES

Jason Bramburger

Limit Cycles - Dynamical Systems | Lecture 21

https://www.youtube.com/watch?v=JVvvjzdip08&t=1s