DOING PHYSICS WITH PYTHON

DYNAMICAL SYSTEMS

GRADIENT SYSTEMS

Ian Cooper
matlabvisualphysics@gmail.com

DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive   **GitHub**

ds2200.py   **ds2201.py**

Gradient systems are differential equations where the vector field is the negative gradient of a potential function $V(\mathbf{x})$, meaning solutions move "downhill" towards a local minimum of that function. In [2D] the system equation is

$$\dot{\mathbf{x}} = -\nabla V(\mathbf{x}) \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -\begin{pmatrix} \partial V / \partial x \\ \partial V / \partial y \end{pmatrix}$$

The solutions (x,y) are fixed points (sinks, sources, or saddles). Orbits or limit cycles do not exist.

The potential $V(\mathbf{x})$ defines a scalar field and the velocity of the system defines a vector field $\dot{\mathbf{x}} = d\mathbf{x} / dt = -\nabla V(\mathbf{x})$, the vector field is the negative gradient of a potential function. The gradient points in the direction of the steepest ascent of the scalar field.

The system moves in the direction opposite to the gradient, which is the direction of steepest descent and solution trajectories are always perpendicular (orthogonal) to the contours of the potential function. Gradient systems do not have closed orbits or cycles.

The potential function $V(\mathbf{x})$ acts as a Lyapunov function, as its value along any trajectory decreases until a fixed point is reached. If a trajectory is bounded, it must eventually approach a state where the gradient is zero at a fixed point (critical point) since

$$\frac{dV(\mathbf{x})}{dt} = \frac{dV}{dx}\frac{d\mathbf{x}}{dt} = \nabla V \bullet \dot{\mathbf{x}} = -\left|\nabla V\right|^2 \le 0 \qquad \dot{\mathbf{x}} = -\nabla V$$

$\dot{V} = 0$ only if $\mathbf{x}$ is at a fixed point, so there cannot be any closed orbits. The flow is always downhill to a local minimum.

Example 1 ds2200.py

Consider the two-dimensional gradient system

$$V(x, y) = x^2 + y^2 + x\,y$$

Then,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -\begin{pmatrix} \partial V / \partial x \\ \partial V / \partial y \end{pmatrix} = -\begin{pmatrix} 2\,x + y \\ 2\,y + x \end{pmatrix}$$

The solution spirals towards the Origin (0, 0) which is a stable sink and a minimum of $V(x, y)$.

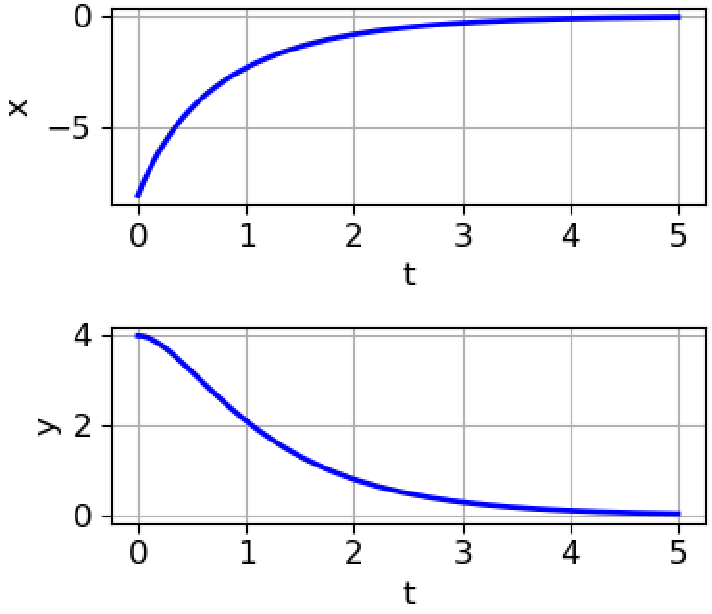

Fig. 1.1. Time evolution of system. The trajectories converges to the stable fixed point at the Origin (0, 0).

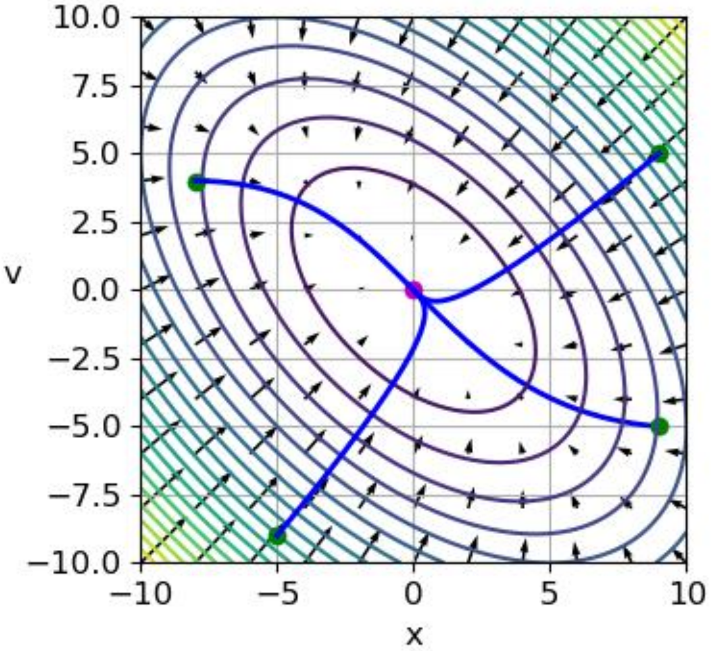

Fig. 1.2. Phase portrait showing the contour lines for the potential. All trajectories converge to the Origin (0, 0). A trajectory is perpendicular to the contour lines.

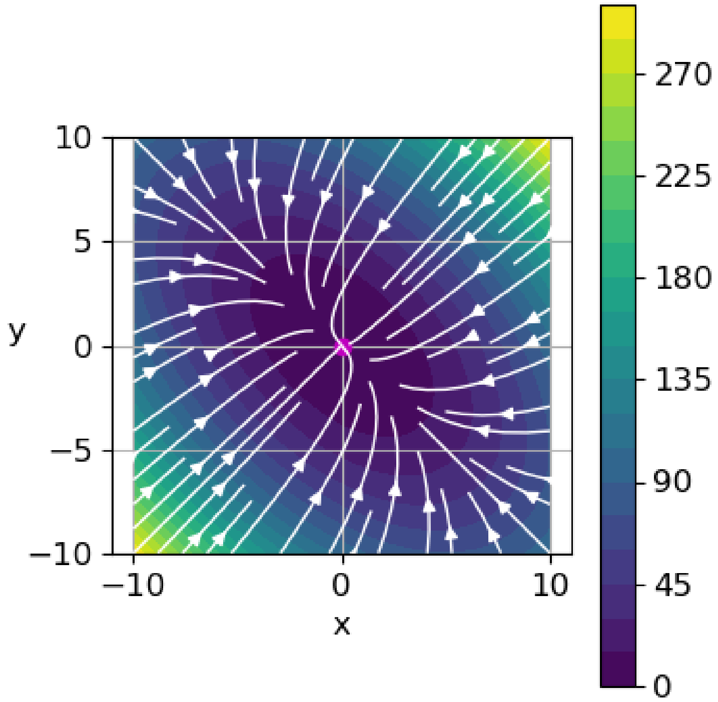

Fig. 1.3. Phase portrait (streamplot). All trajectories go downhill to the local minimum.

Example 2 **ds2201.py**

Consider the potential function V and gradient system

$$V = x^3 + y^3 - 12\,x\,y$$

$$\partial V / \partial x = 3x^2 - 12\,y \qquad \partial V / \partial y = 3y^2 - 12\,x$$

$$\dot{x} = -\left(3x^2 - 12\,y\right) \qquad \dot{y} = -\left(3y^2 - 12\,x\right)$$

Fixed points:

(0, 0) saddle

(4, 4) stable node

x nullcline: $y = x^2 / 4$ y nullcline: $x = y^2 / 4$

Figure 2.1 shows the phase portrait as a streamplot. A local minimum is at (4,4) and this produces a trapping region surrounding it where all trajectories converge to stable node at (4,4). Trajectories with initial conditions not in the trapping region all diverge rapidly to infinity

$$t \to \infty \quad \Rightarrow \quad x \to -\infty \quad y \to -\infty$$

The Origin (0,0) is a saddle node since trajectories are attracted to it from afar but are repelled near it.

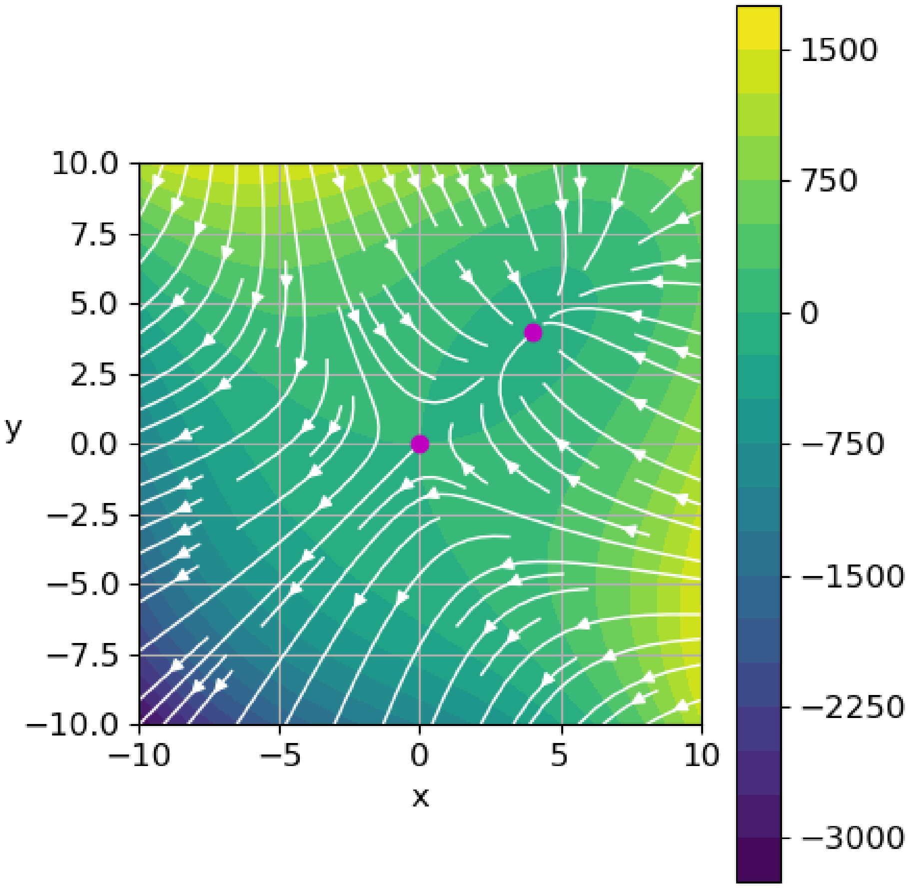

Fig. 2.1. Phase portrait. The potential $V(x,y)$ is given by the colour. The magenta dots are the two fixed points (4,4) which is a stable node and (0,0) is a saddle node.

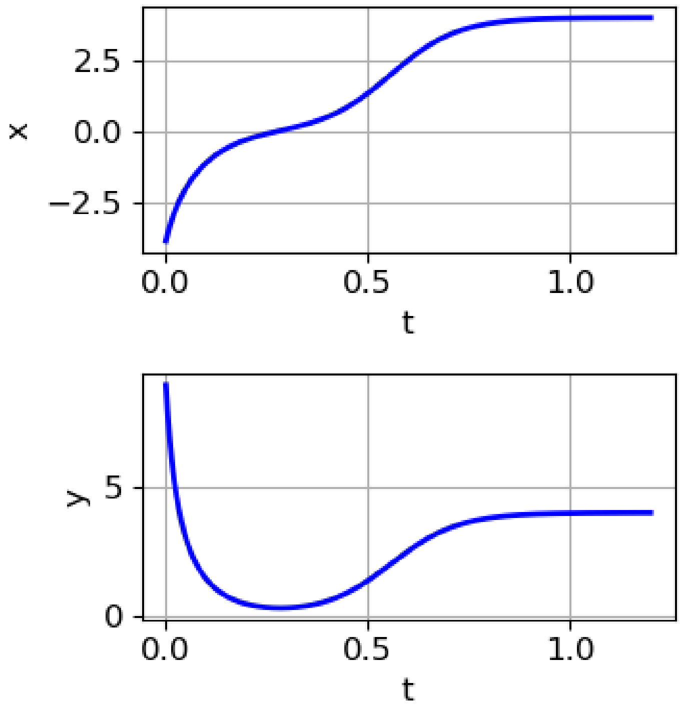

Fig. 2.2A.   Trajectories converge to the stable node (4,4).

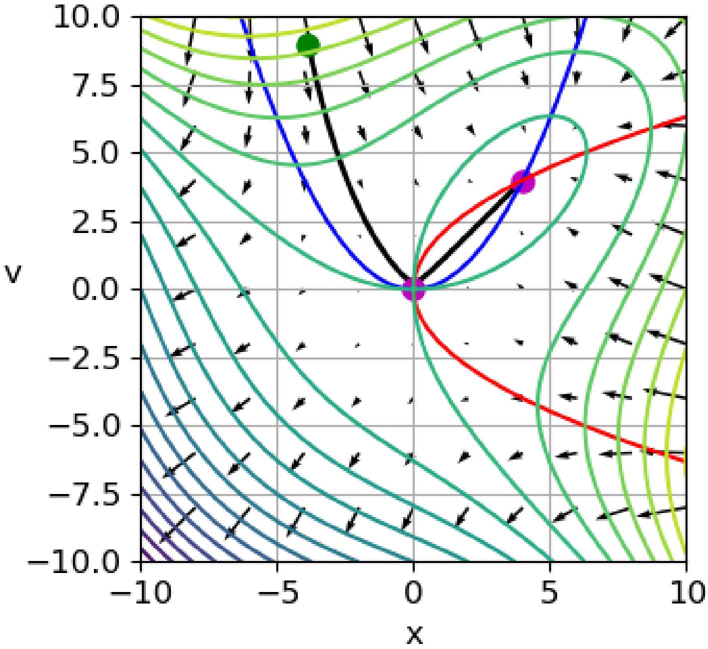

Fig. 2.2B. Contour plot of potential. Blue curve is the *x* nullcline and red curve is the *y* nullcline. Initial conditions (**-3.87**, 9).

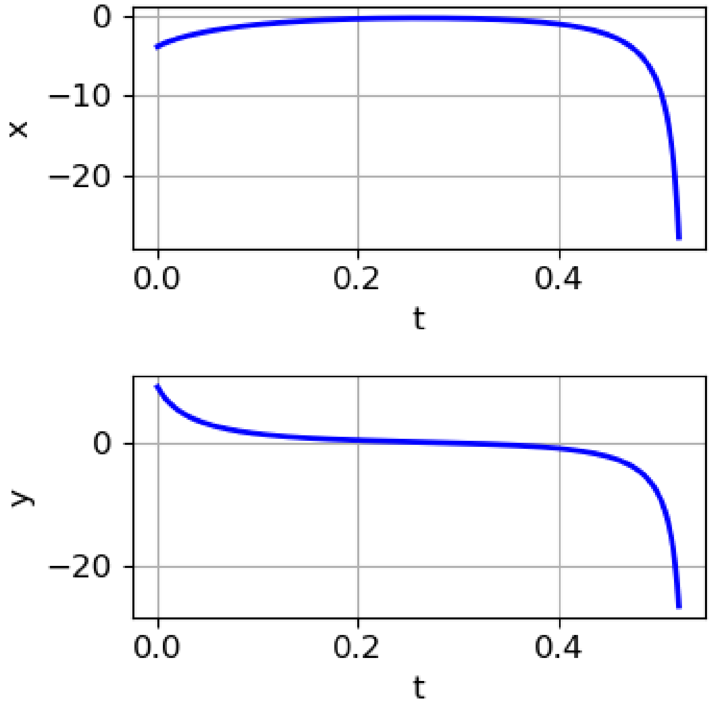

Fig. 2.3A. Trajectories diverge from the saddle node (0,0).

$$t \to \infty \quad \Rightarrow \quad x \to -\infty \quad y \to -\infty$$

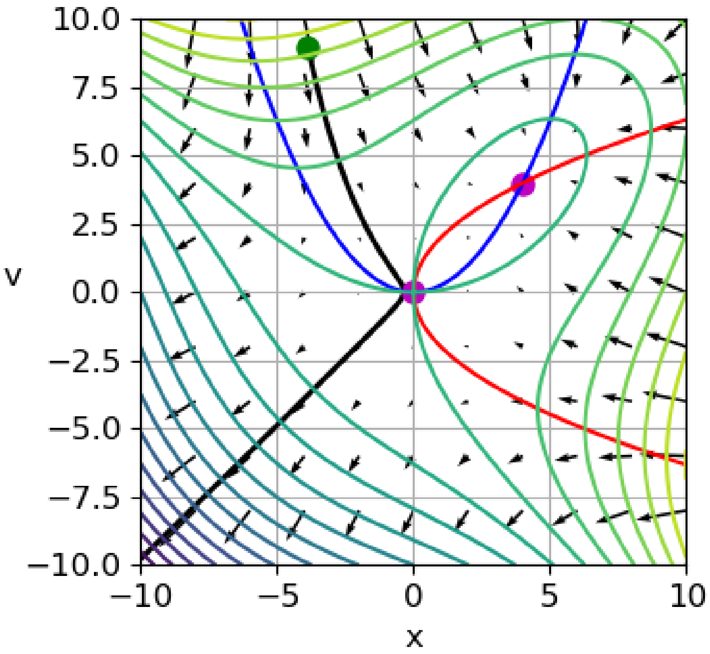

Fig. 2.3B. Contour plot of potential. Blue curve is the *x* nullcline and red curve is the *y* nullcline. Initial conditions (**-3.88**, 9).

Comparing figures 2.2 with 2.3, you will notice that a very small change in the initial conditions has a dramatic effect upon the trajectories of the system.

(-3.88, 9) $\quad t \to \infty \;\Rightarrow\; x \to -\infty \quad y \to -\infty$

(-3.87, 9) $\quad t \to \infty \;\Rightarrow\; x \to +4 \quad y \to +4$

There are other dynamical systems that also do not have a limit cycle. For example, systems which have an energy dissipation term.

REFERENCES

Jason Bramburger

Gradient Systems - Dynamical Systems | Lecture 22

https://www.youtube.com/watch?v=imJHnWrXWDg&t=653s

Bill Kinney

Gradient System Defined by Gradient Vector of Potential Function (also consider Hamiltonian System)

https://www.youtube.com/watch?v=eAQEKLRmOH4