DOING PHYSICS WITH PYTHON

DYNAMICAL SYSTEMS

SADDLE NODE BIFURCATIONS IN PLANAR SYSTEMS

Ian Cooper
matlabvisualphysics@gmail.com

DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive **GitHub**

ds2500.py

A saddle node bifurcation in planar systems occurs when fixed points are created or destroyed as a parameter changes. This is a fundamental type of bifurcation (fold, blue-sky, tangent, or limit point bifurcation) that happens when the system's dynamics change qualitatively at a specific parameter value. In a planar system, this bifurcation is visualized when the nullclines of the system become tangent to each other leading to the collision and annihilation or creation of these fixed points.

As a parameter crosses a critical value, two fixed points can emerge from nowhere or collide and disappear. For a slight change in the parameter value, the system's overall behaviour changes significantly.

Example

Consider the [2D] system

$$\dot{x} = -a\,x + y \qquad \dot{y} = \frac{x^2}{1+x^2} - b\,y \quad a,\, b > 0$$

This system was proposed by Griffth in 1971 as a model for genetic control where x is the concentration of a gene protein and y a concentration of mRNA.

x nullcline $\quad \dot{x} = 0 \qquad y = -a\,x$

y nullcline $\quad \dot{y} = 0 \qquad y = \dfrac{x^2}{b\left(1+x^2\right)}$

The Origin (0, 0) is always a fixed point.

Other fixed points (x_e, y_e) can be found from the intersection of the two nullclines

$$-a\,x_e = \frac{x_e^{\,2}}{b\left(1 + x_e^{\,2}\right)}$$

$$a\,b\,x_e^{\,2} + x_e + a\,b = 0$$

$$x_e = \frac{1 \pm \sqrt{1 - 4\,a^2 b^2}}{2\,a\,b}$$

Take *b* as a constant and *a* as the bifurcation parameter:

$a < 1/(2b)$ two extra fixed points (3 fixed points)

$$x_e = \frac{1 + \sqrt{1 - 4\,a^2 b^2}}{2\,a\,b} \qquad y_e = a\,x_e$$

$$x_e = \frac{1 - \sqrt{1 - 4\,a^2 b^2}}{2\,a\,b} \qquad y_e = a\,x_e$$

$a = 1/(2b) = a_c$ a_c critical value of the bifurcation parameter

one extra fixed point (2 fixed points)

$$x_e = \frac{1}{2\,a\,b} \qquad y_e = \frac{1}{2b}$$

$a > 1/(2b)$ Origin is the only fixed point (1 fixed point)

Finding the eigenvalues of the Jacobian matrix can help determine the stability of the fixed points.

$$\mathbf{J}(x_e, y_e) = \begin{pmatrix} \partial f / \partial x & \partial f / \partial y \\ \partial g / \partial x & \partial g / \partial y \end{pmatrix}_{|x = x_e, y = y_e}$$

$$\mathbf{J}(x_e, y_e) = \begin{pmatrix} -a & 1 \\ 2x_e / (1 + x_e^{\,2})^2 & -b \end{pmatrix}$$

For the fixed point at the Origin (0, 0), the eigenvalues are (-a, -b) and since $a > 0$, and $b > 0$, the eigenvalues are both negative. Hence, the Origin (0, 0) is always a stable fixed point.

For other fixed points, the eigenvalues can be computed using the Python function **eig**.

SIMULATIONS

For all simulation $b = 1$ and a is the bifurcation parameter with a critical value $a_c = 0.50$. Figure 1 shows a bifurcation diagram for the fixed points (x_e, y_e) as a function of the bifurcation parameter a.

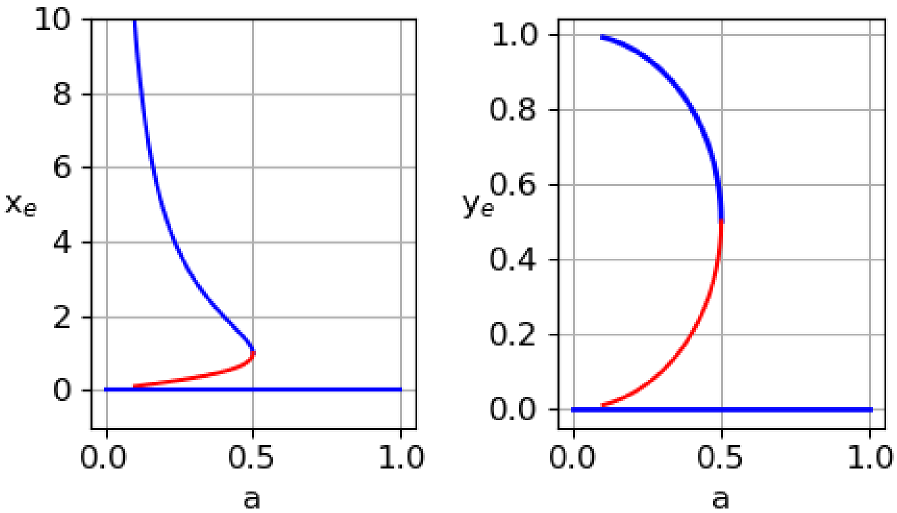

Fig. 1. Bifurcation diagram. The blue curves are stable fixed points and the red curves are for saddle nodes. $a < a_c$ there are three fixed points and for $a > a_c$ there is only one fixed point.

Depending on the value of the bifurcation parameter a, there are either one, two or three fixed points. The Origin (0, 0) is always a stable fixed point. At the bifurcation point, $a_c = 0.50$ there are two fixed points. There is a change in the number of fixed points and changes in stability as the value of a crosses the critical value a_c.

$a = 0.6 > a_c = 0.5$

Figure 2 shows the time evolution of x and y and the phase portraits as a quiver plot and a streamplot (blue line is the x nullcline, the red line is the y nullcline and the green line the trajectory). There is only one fixed point, the Origin (0, 0) which is stable. The eigenvalues (displayed in the Console Window) of the Jacobian matrix are

$$\lambda_1 = -0.60 \quad \lambda_2 = -1.00 \;\Rightarrow\; \text{stable node}$$

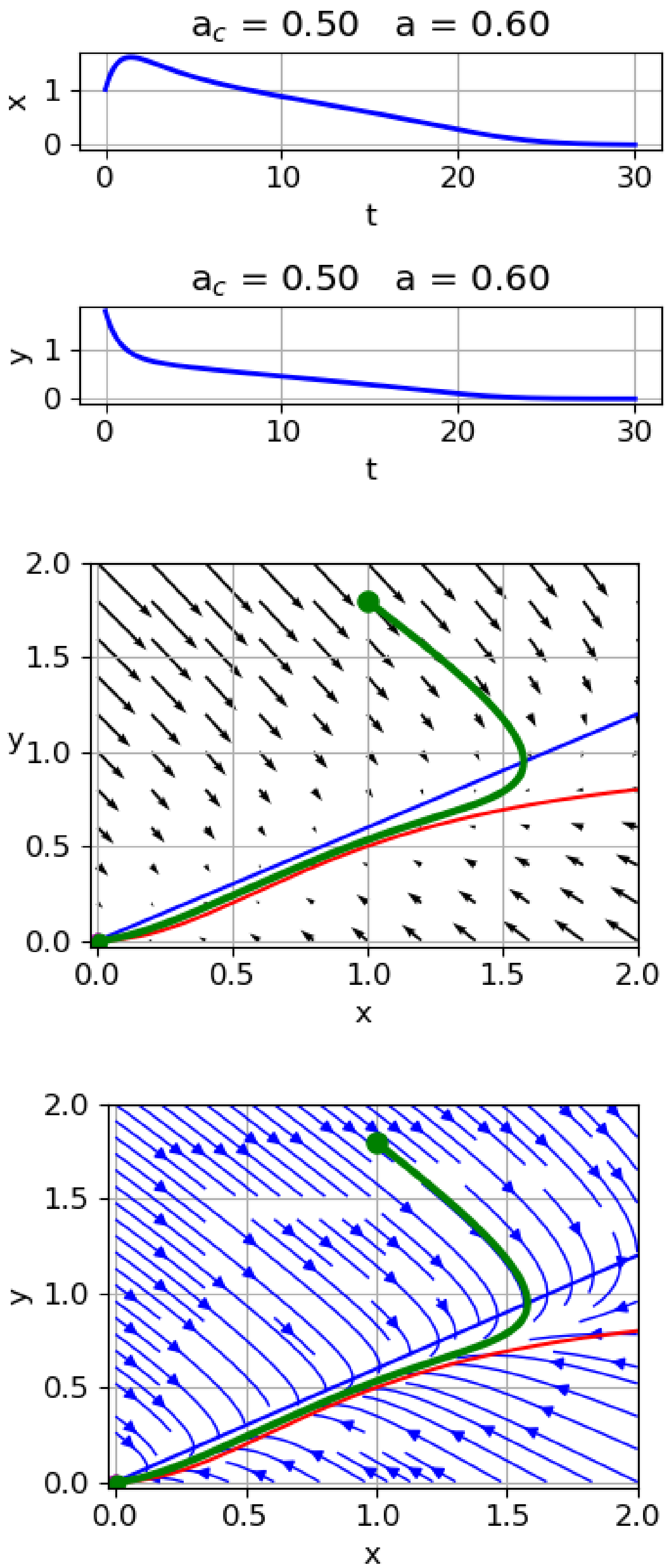

Fig. 2. $a = 0.6 > a_c = 0.5$

$a = 0.5 = a_c$

When the value of a decreases from 0.60 to 0.050 there is a creation of another fixed point. The two fixed points are the Origin (0, 0) which is stable and (1.00, 0.50) which is also stable and its eigenvalues are

$$\lambda_1 = 0 \quad \lambda_2 = -1.50 \Rightarrow \text{? node}$$

A dynamical system with one zero eigenvalue and one negative eigenvalue is non-hyperbolic and marginally stable, meaning it is stable but not asymptotically stable. As the zero eigenvalue means the stability of the equilibrium cannot be determined from the eigenvalues alone. By examination of trajectories in the phase portrait, we can conclude that (1.00, 0.50) is a stable fixed point. As shown in figure 3, depending upon the initial conditions, the trajectory is attracted to the fixed point (0, 0) or the fixed point (1.00, 0.50).

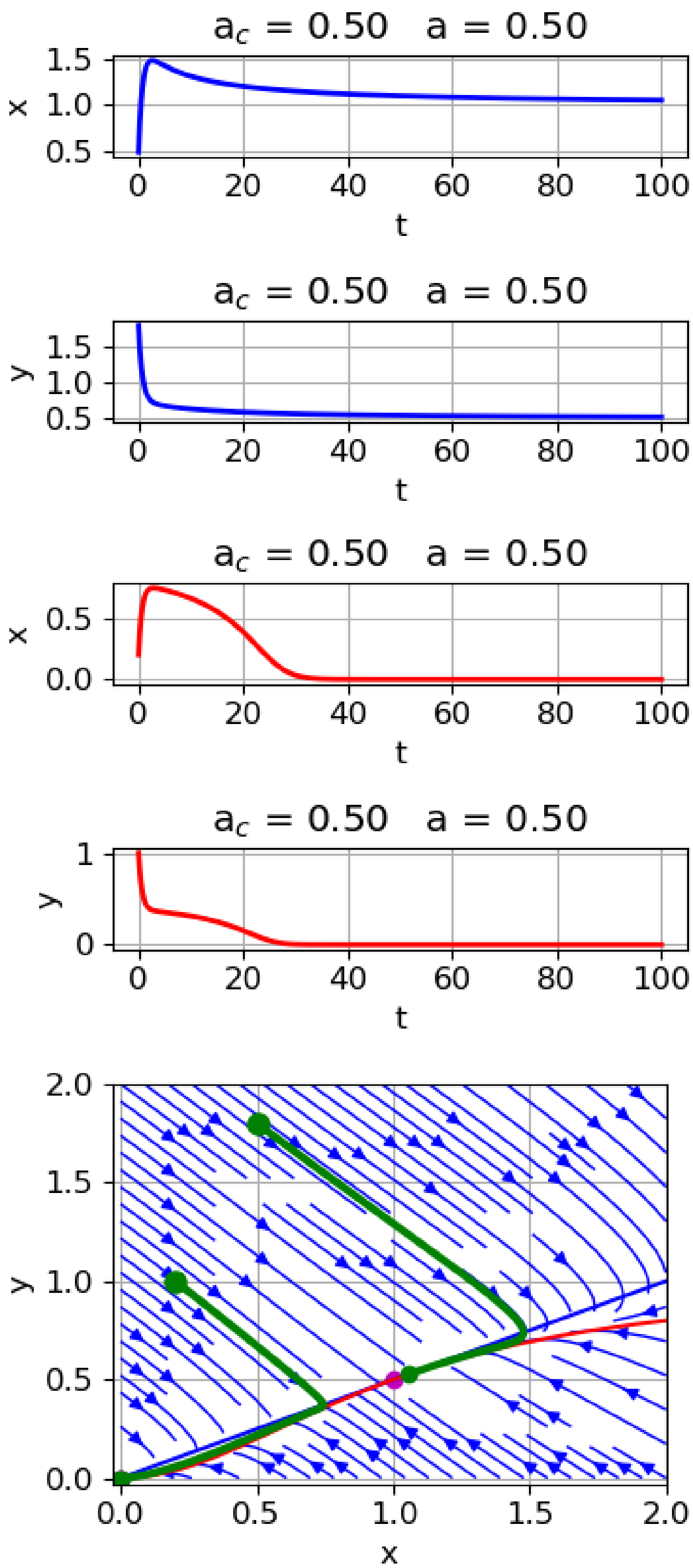

Fig. 3. ***a* = 0.5 = a_c** Depending upon the initial conditions, the trajectory is attracted to the fixed point (0, 0) or the fixed point ((1.00, 0.50).

$a = 0.45 < a_c = 0.50$

There are three fixed points

(0, 0) stable

(0.63, 0.28) $\lambda_1 = 0.13 \quad \lambda_2 = -1.58 \Rightarrow$ saddle node

(1.60, 0.72) $\lambda_1 = -0.15 \quad \lambda_2 = -1.30 \Rightarrow$ stable

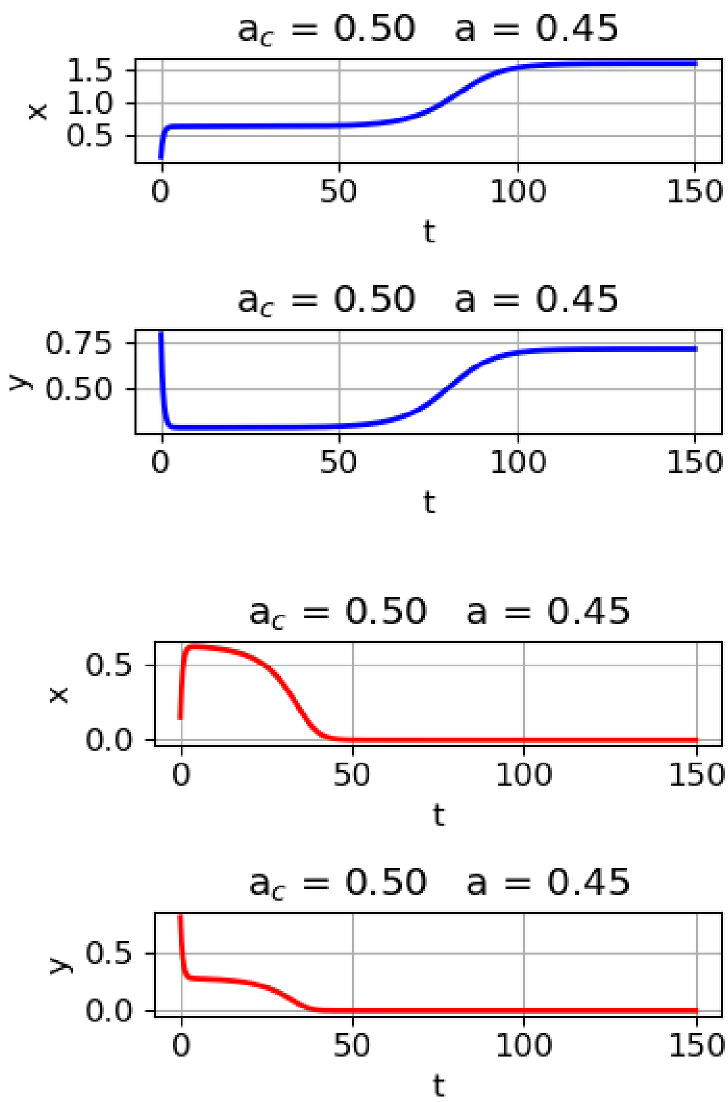

Fig. 4A. $x(0) = \mathbf{0.16}, \; y(0) = 0.80$

$x(0) = \mathbf{0.15}, \; y(0) = 0.80$

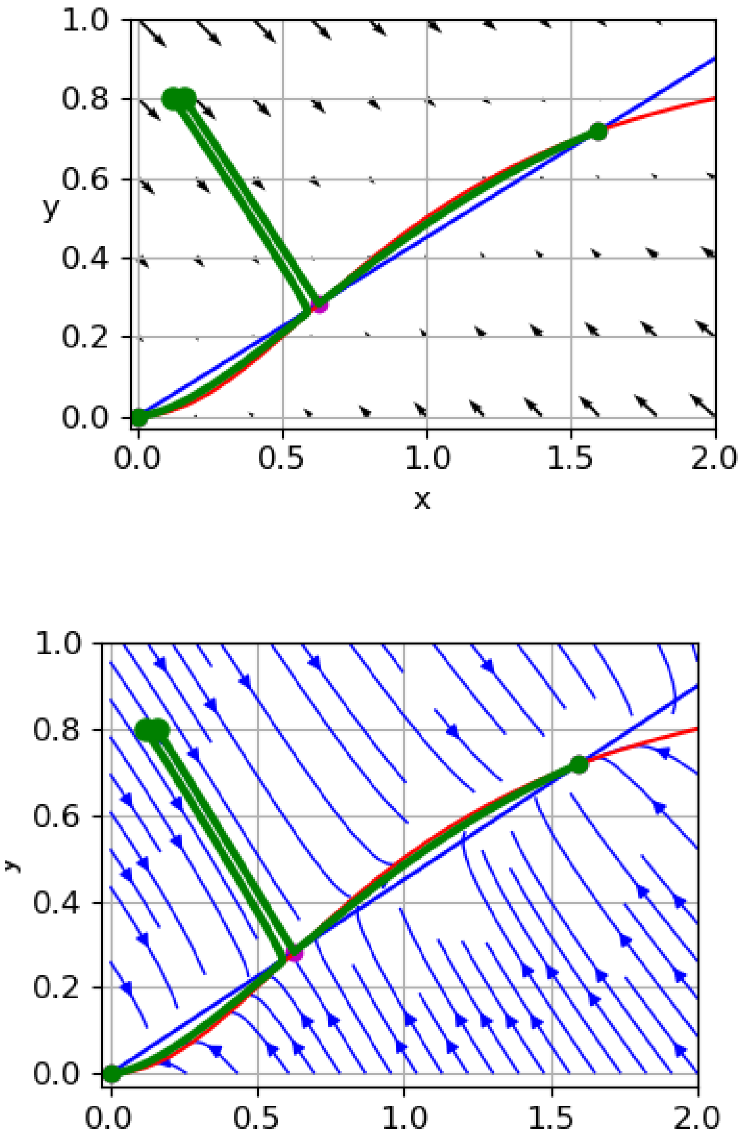

Fig. 4B. The fixed point (0.63, 0.28) is a saddle node while the other two fixed points (0, 0) and (1.60, 0.72) are stable nodes. The streamplot clearly shows the saddle node: far from the saddle, the trajectory is attracted, but close to the saddle the trajectory is repelled.

REFERENCES

Jason Bramburger

Bifurcations in Planar Systems - Dynamical Systems | Lecture 25

https://www.youtube.com/watch?v=b_s4pcx-YoQ&t=1255s