DOING PHYSICS WITH PYTHON

DYNAMICAL SYSTEMS

PITCHFORK BIFURCATIONS IN PLANAR SYSTEMS

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ds2501.py

A pitchfork bifurcation in a planar system occurs when a small change in a parameter causes a single stable equilibrium point to split into two new stable equilibrium points, or vice versa, with the original point becoming unstable. This phenomenon is named for its characteristic "pitchfork" shape in a bifurcation diagram, where the stable branches split from the original point. Planar systems can exhibit both supercritical pitchfork bifurcations (where the split creates new stable points from a stable parent) and subcritical pitchfork bifurcations (where the split creates new unstable points from an unstable parent), often arising in systems with an odd symmetry.

Consider the dynamical system

$$(1) \qquad \dot{x} = -a\,x + y + \sin(x) \qquad \dot{y} = x - y$$

where a is the bifurcation parameter.

y nullcline

$$(2) \qquad \dot{y} = 0 \qquad y = x$$

x nullcline

$$(3) \qquad \begin{array}{ll} \dot{x} = 0 & a\,x + y + \sin(x) = 0 \\ & x\left(a+1\right) + \sin(x) = 0 \end{array}$$

The Origin (0, 0) is always a fixed point.

Other fixed points (x_E, y_E) can be found from the intersection of the two nullclines.

Let

$$F(x) = x\left(a+1\right) + \sin(x)$$

then

$$(4) \qquad F(x_E) = x_E\left(a+1\right) + \sin(x_E) = 0 \qquad y_E = x_E$$

The fixed points (x_E, y_E) are found by solving equation 4. This equation for finding the fixed points is more complicated than it looks. The number of fixed points is highly dependent upon the value of the bifurcation parameter a and the range of x values as shown in figure 1.

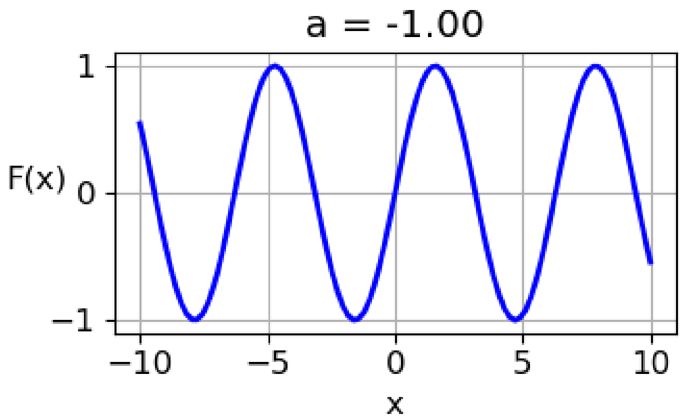


Fig. 1A. $a = -1 \Rightarrow \; F(x) = \sin(x) \quad F(x_E) = 0 \quad \sin(x_E) = 0$

There is an infinite number of fixed points. The Origin is unstable.

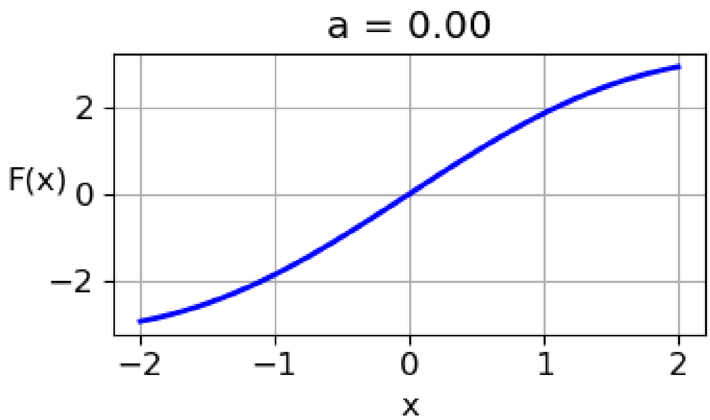


Fig. 1B. $a = 0 \Rightarrow \; F(x) = x + \sin(x) \quad F(x_E) = 0 \quad x_E = -\sin(x_E)$

$x_E = 0 \quad y_E = 0$

The Origin is the only fixed point and is unstable.

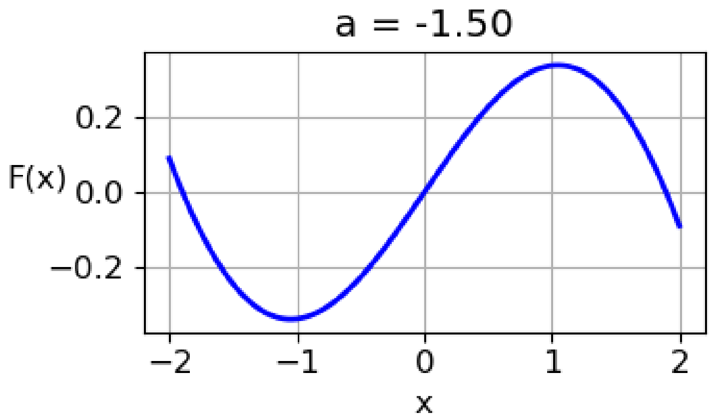


Fig. 1C. $a = -1.5$ The are three fixed points.

(0, 0) unstable

(-1.90,-1.90) stable

(+1.90, +1.90) stable

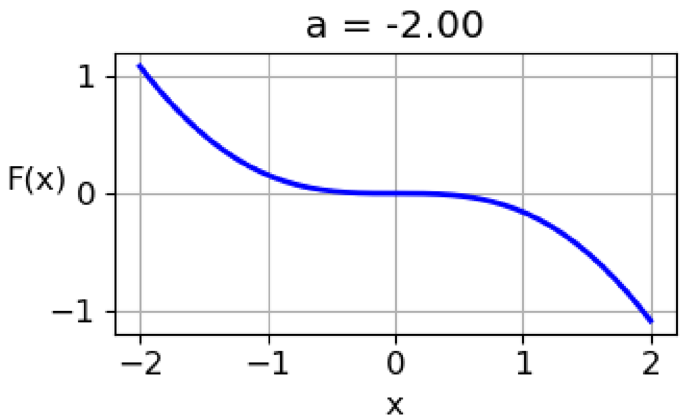


Fig. 1D. a = -2.0 There is one fixed point.

(0, 0) stable

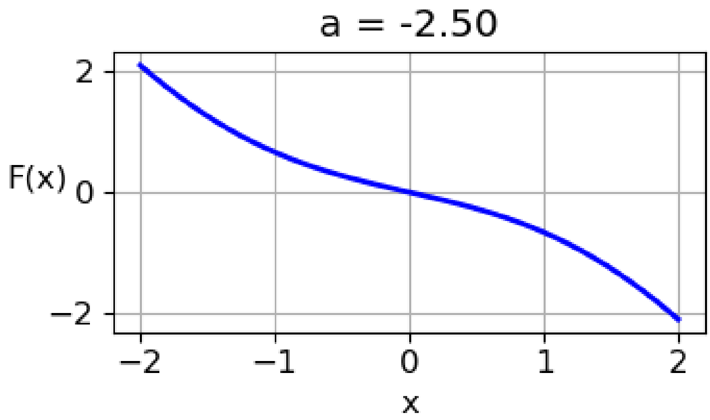


Fig. 1E. $a = -2.5$ There is one fixed point.

(0, 0) stable

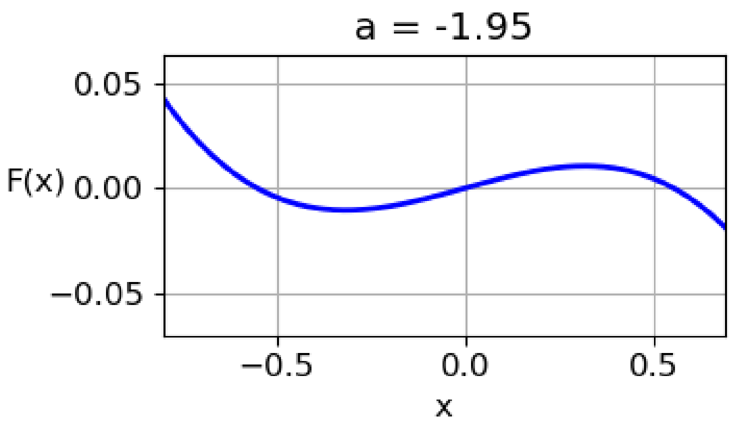


Fig. 1F. $a = -1.95$ The are three fixed points.

(0, 0) unstable

(- 0.55, - 0.55) stable

(+ 0.55, + 0.55) stable

To understand the pitchfork bifurcation using the Python Code **ds2501.py**, the bifurcation parameter a is restricted to the range -3 to +2 and the x range to -2 to +2.

Finding the eigenvalues of the Jacobian matrix can help determine the stability of the fixed points.

$$\mathbf{J}(x_e, y_e) = \begin{pmatrix} \partial f / \partial x & \partial f / \partial y \\ \partial g / \partial x & \partial g / \partial y \end{pmatrix}_{|x=x_e, y=y_e}$$

$$\mathbf{J}(x_e, y_e) = \begin{pmatrix} a + \cos(x) & 1 \\ 1 & -1 \end{pmatrix}$$

The Origin (0, 0) is always a fixed point and the dependence of the Origin's eigenvalues is shown in figure 2.

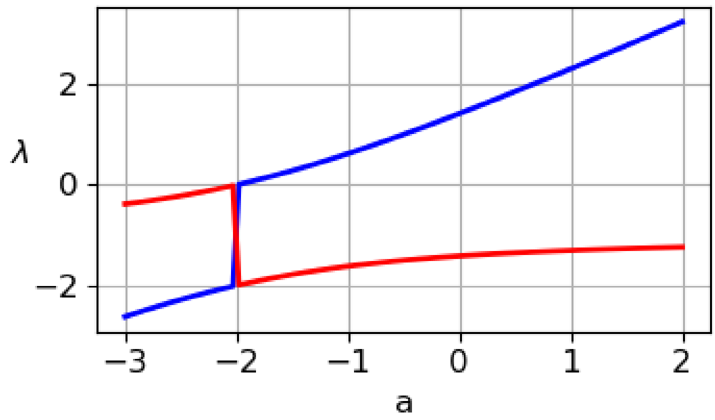
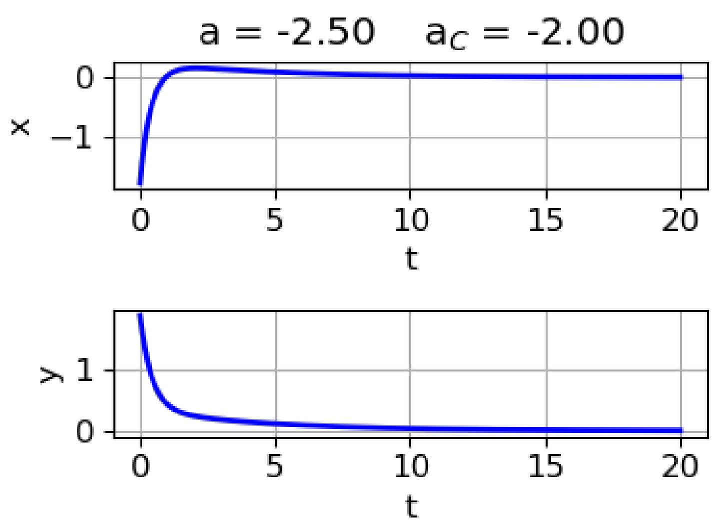
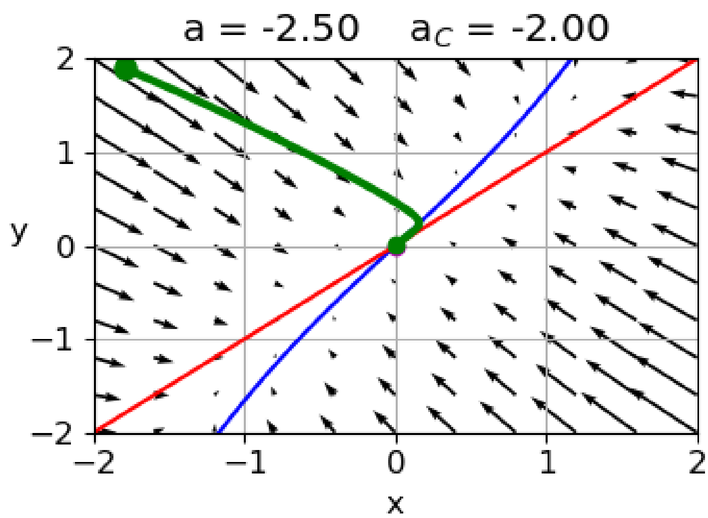


Fig. 2. The red and blue curves give the values for the two eigenvalues. The critical value of the bifurcation parameter is a_C = -2.

For a < -2, both eigenvalues are negative. Hence, the Origin (0, 0) is a stable fixed point. A dynamical system with one positive and one negative eigenvalue is an unstable saddle point. Trajectories will diverge along the eigenvector associated with the positive eigenvalue and converge along the eigenvector associated with the negative eigenvalue. The system is unstable because any small perturbation in the direction of the positive eigenvalue will cause the state to move away from the origin. Therefore, when $a > -2$, the Origin is an unstable saddle.

Figure 3 shows a series of plots when the fixed point at the Origin (0, 0) is stable ($a < a_C = -2$) and all trajectories are attracted to it. The nullclines only intersect at (0, 0) and so the Origin is the only fixed point of the system





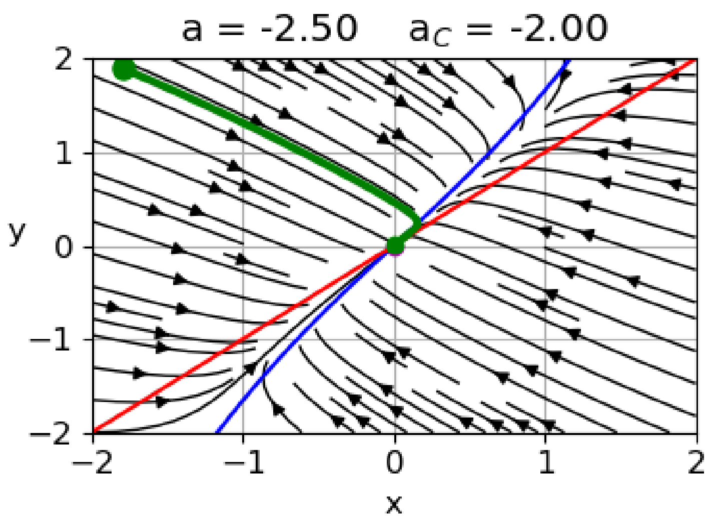


Fig. 3. a = -2.50 < a_c = -2.00. In the phase portraits, the blue line is the x nullcline and the red line is the y nullcline.

Figure 4 shows the pitchfork bifurcation behaviour occurring when $a > a_c$ = -2.00. For a = -1.60 > a_c, the Origin becomes a saddle node and two stable nodes are created.

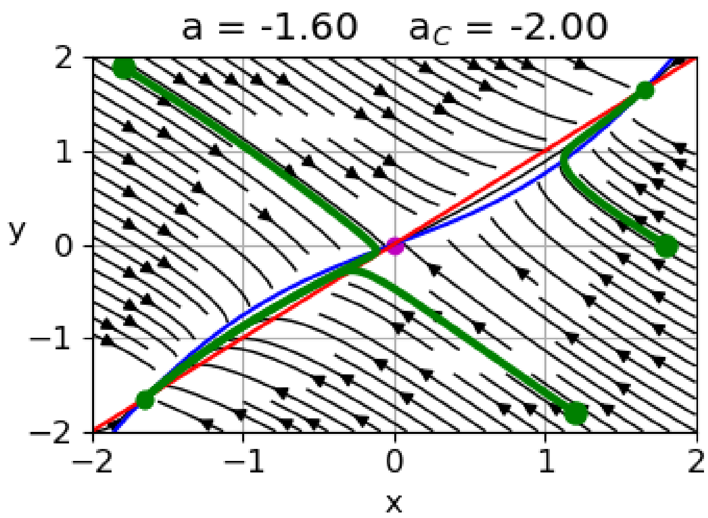


Fig. 4A. The phase portrait shows that there are three fixed point and the streamlines shown their stability: (0, 0) saddle node; (-1.66, -166) stable node; (+1.66, +1.66) stable node.

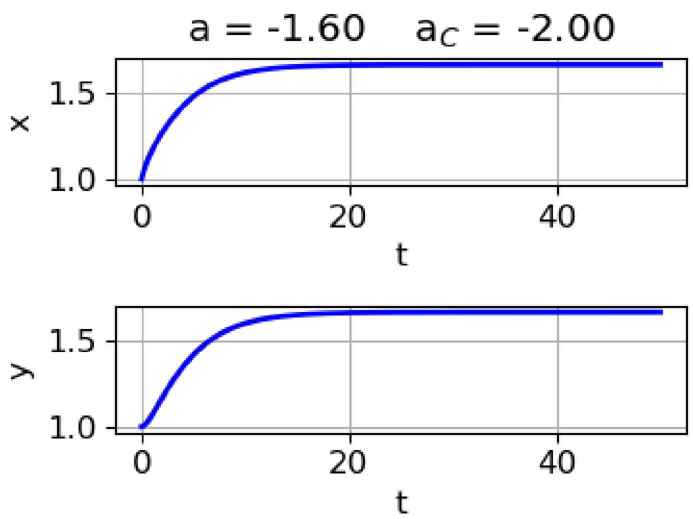


Fig. 4B. Time evolution of a trajectory which is attracted to the stable fixed point (+1.66, +1.66).

The eigenvalues of the Jacobin matrix evaluated at the fixed points can be calculated using the Python function **eig** and the results displayed in the Console Window:

(-1.66, -166) eigenvalues **[-3.121 -0.529]** **stable**

(+1.66, +166) eigenvalues **[-2.402 -0.287]** **stable**

This system is an example of a **supercritical pitchfork bifurcation**.

REFERENCES

Jason Bramburger

Bifurcations in Planar Systems - Dynamical Systems | Lecture 25

https://www.youtube.com/watch?v=b_s4pcx-YoQ&t=1255s