DOING PHYSICS WITH PYTHON

DYNAMICAL SYSTEMS [1D]

Population Dynamics 1

Ian Cooper
matlabvisualphysics@gmail.com

DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive

GitHub

ds25L10.py

Jason Bramburger

Bifurcations in a Model for Insect Outbreak - Dynamical Systems | Lecture 10

https://www.youtube.com/watch?v=ASYXwNCZdds

INTRODUCTION

Population growth models describe how the size of a population changes over time. The two fundamental models are exponential growth, which shows a population growing at an ever-increasing rate with unlimited resources, and the logistic model, a more realistic model where growth slows and levels off at the environment's carrying capacity, *K* due to resource limitations.

In this article we look at an extended logistic model for insect outbreak and study the bifurcations and demonstrate that a saddle-node bifurcation takes place.


The logistic model describes population growth that slows as it approaches a carrying capacity *K*, the maximum population size an environment can sustain. Unlike exponential growth which shows unlimited J-shaped growth, logistic growth produces an S-shaped curve because limited resources create density-dependent factors (like reduced birth rates and increased death rates) that restrict growth as the population size *N* increases. The population eventually plateaus at the carrying capacity *K*, creating a realistic model of population dynamics limited by environmental constraints.

The Logistic Equation

The growth rate of the population in the logistic model is described by the ODE

$$(1) \qquad \frac{dN}{dt} = \dot{N} = r\,N\left(1 - \frac{N}{K}\right)$$

$\dot{N}$ is the rate of change of the population N with respect to time t

r is the growth rate

N is the population size

K is the carrying capacity

We can introduce the concept of predation into our population dynamics. Predation influences the rate of change of a population where predators decrease prey numbers and, in turn, their own populations are affected by prey availability.

$$(2) \qquad \dot{N} = r\,N\left(1 - \frac{N}{K}\right) - P$$

where $P(N)$ is the predation term. For example, N is the number of insects and $P(N)$ is the rate at which birds eat the insects. We will use the function given by equation 3 for the predation

$$(3) \qquad P = \frac{B\,N^2}{A^2 + N^2}$$

Where A and B are constants. Combining equations 1 and 3 we obtain the ODE for the dynamics of the insect population

(4) $$\dot{N} = r\,N\left(1 - \frac{N}{K}\right) - \frac{B\,N^2}{A^2 + N^2}$$

The model governed by equation 4 is a poor one, since there is no dependence on the bird population or any term for the interaction between insect and bird numbers. As a consequence, the initial population can be zero (extinction) or reach an equilibrium state

$$N(0) \to 0 \quad \text{or} \quad N(0) \to N_e \neq 0$$

In Jason's video equation 4 is expressed in a dimensionless formulation which reduces the number of constants from 4 to 2. However, this is not necessary when you can solve the ODE numerically. So, the Python Code **ds25L10.py** solves equation 4 where you need to input the values for the constants *r*, *K*, *A* and *B*.

The fixed points N_e (equilibrium populations) of the system are determined from equation 4 when $\dot{N} = 0$.

(5) $$r\,N\left(1 - \frac{N}{K}\right) - \frac{B\,N^2}{A^2 + N^2} = 0$$

It is obvious that $N_e = 0$ is one fixed point. The other fixed points are found graphically by plotting N *vs* $\dot{N}$ and determining the zero crossing of the function.

```
x = linspace(0,1.1*K,num)
xDot = r*x*(1 - x/K) - B*x**2/(A**2 + x**2)
# Find zeros in Ndot
Q = np.zeros(2); p = 0
for c in range(num-2):
   q = xDot[c]*xDot[c+1]
   if q <= 0:
     Q[p] = c
     p = int(p+1)
QI = Q.astype(int)
xZ = x[QI]        # Zeros for xDot
```

The constants r, A and B are bifurcation parameters. The number of fixed points depends upon the values of r, A and B and bifurcations occur at critical values of r, A and B.

Zero population $N = 0$ means that the population is extinct and there can be no growth in the population. However, for a very small population, the fixed point $N_e = 0$ may be stable or unstable depending on the values of r, A and B.

The condition for the fixed points is given by equation 5, which can be expressed $\left(N_e \neq 0\right)$as

$$r\left(1-\frac{N_e}{K}\right)=\frac{B\,N_e}{A^2+N_e^{\ 2}}$$

(6)

$$y_1=r\left(1-\frac{N_e}{K}\right)\quad y_2=\frac{B\,N_e}{A^2+N_e^{\ 2}}\quad y_1=y_2$$

So, when plotted, the intersection of the two functions y_1 and y_2 give the values for the fixed points when $N_e \neq 0$.

For the simulations shown below the values of *r*, *K* and *B* are kept constant. The constant *A* is varied and is the bifurcation parameter for the simulations.

SIMULATIONS

Simulation 1 $r = 1.5$ $K = 1.0$ $\mathbf{A = 0.100}$ $B = 0.300$

4 fixed points

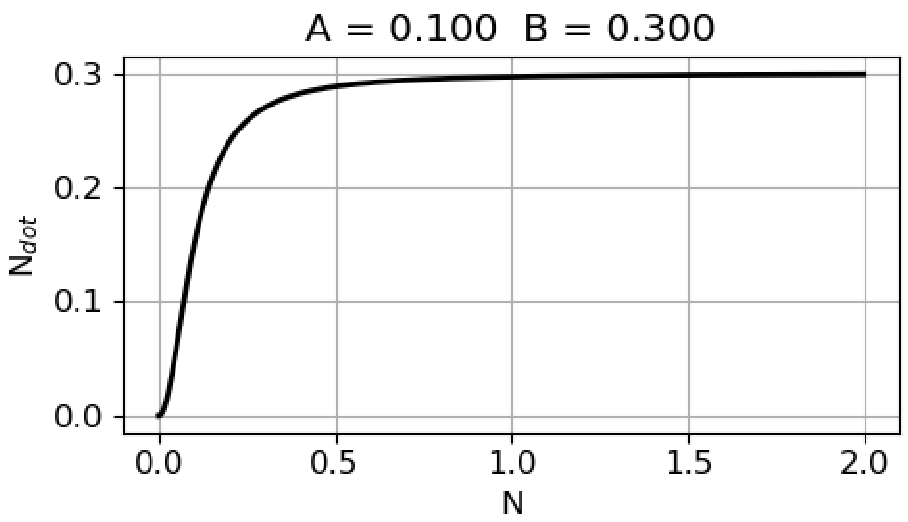

Fig. 1A. Predation curve (equation 3).

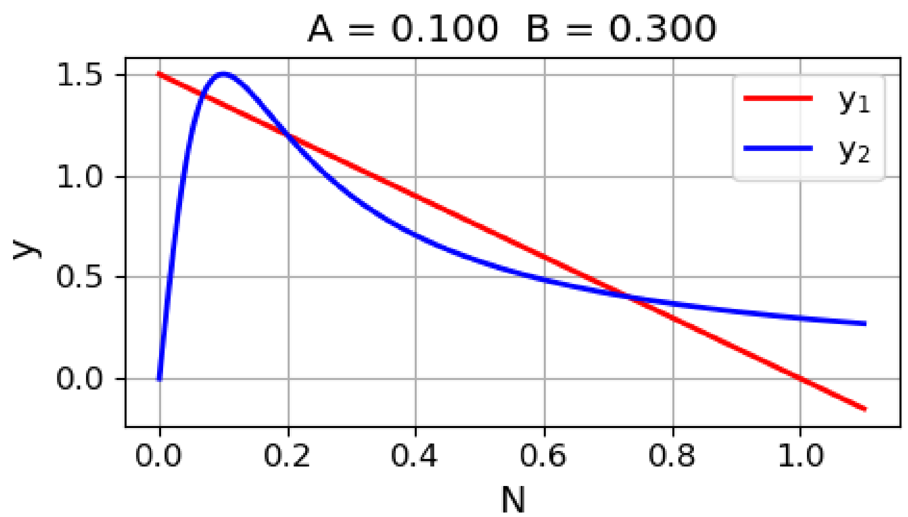

Fig. 1B. Equation 6 plots.

$$N \neq 0 \quad \dot{N} = N\left(y_1 - y_2\right)$$

$y_1 > y_2 \Rightarrow \dot{N} > 0$ flow to the right $\rightarrow$

$y_1 < y_2 \Rightarrow \dot{N} < 0$ flow to the left $\leftarrow$

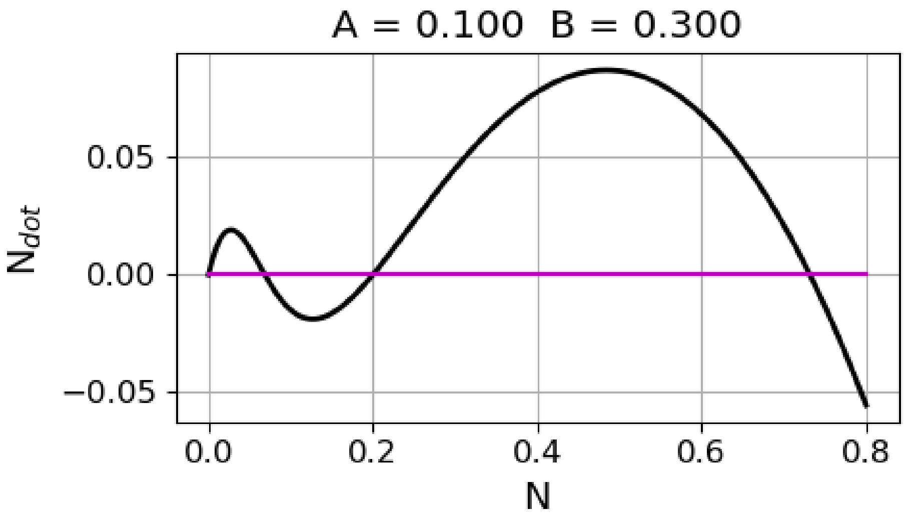

Fig. 1C. Equation 4 plot. There are four fixed points for $A = 0.100$ and $B = 0.300$.

$\boldsymbol{x_e = 0}$	**unstable**	(positive slope)
$x_e = 0.0681$	stable	(negative slope)
$\boldsymbol{x_e = 0.1996}$	**unstable**	(positive slope)
$\boldsymbol{x_e = 0.7311}$	**stable**	(negative slope)

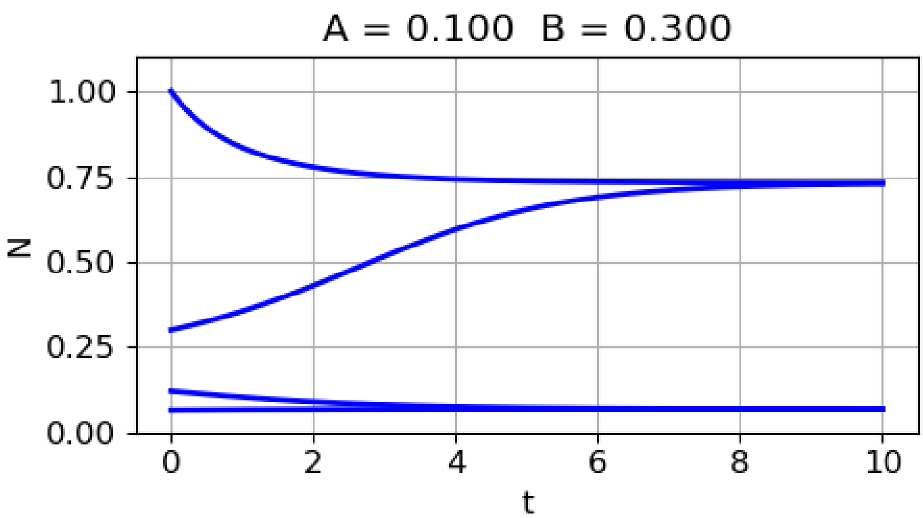

Fig 1D. Population trajectories. For different initial conditions, all trajectories converge to a stable fixed point.

There are two stable fixed points. If x_e **= 0.7311** there is an **outbreak** in the insect population. But if x_e = 0.681 there is no outbreak and we have only a small insect population. Whether there is an outbreak in the insect population depends upon the initial number of insects.

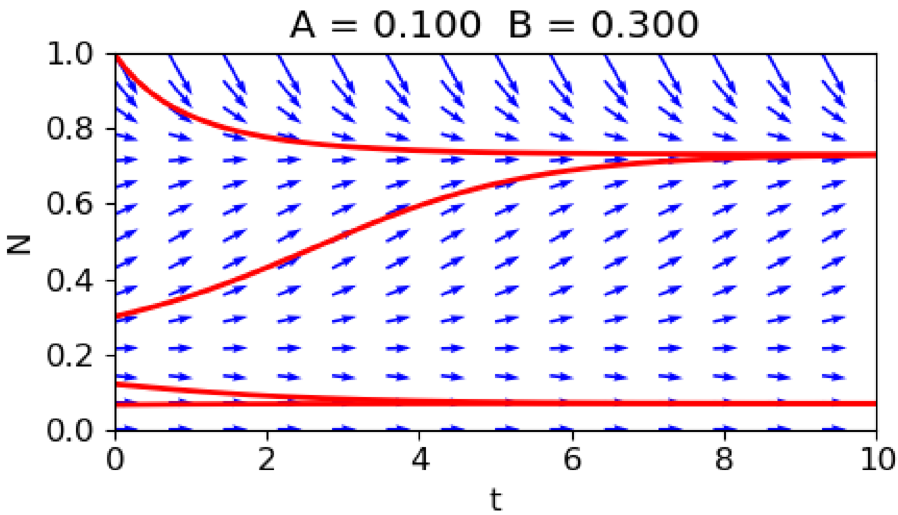

Fig. 1E. Vector field quiver plot and insect population trajectories. The trajectories are repelled from an unstable fixed point and attracted to a stable fixed point.

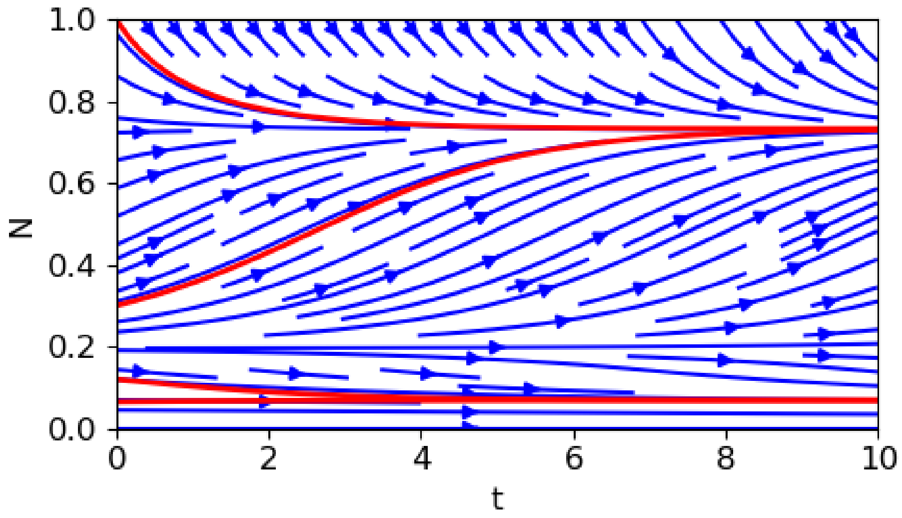

Fig. 1F. Vector field streamplot plot and insect population trajectories. The “flow” is away from an unstable fixed point and towards a stable fixed point.

Simulation 2 $r = 1.5 \quad K = 1.0 \quad \boldsymbol{A = 0.114} \quad B = 0.300$

3 fixed points

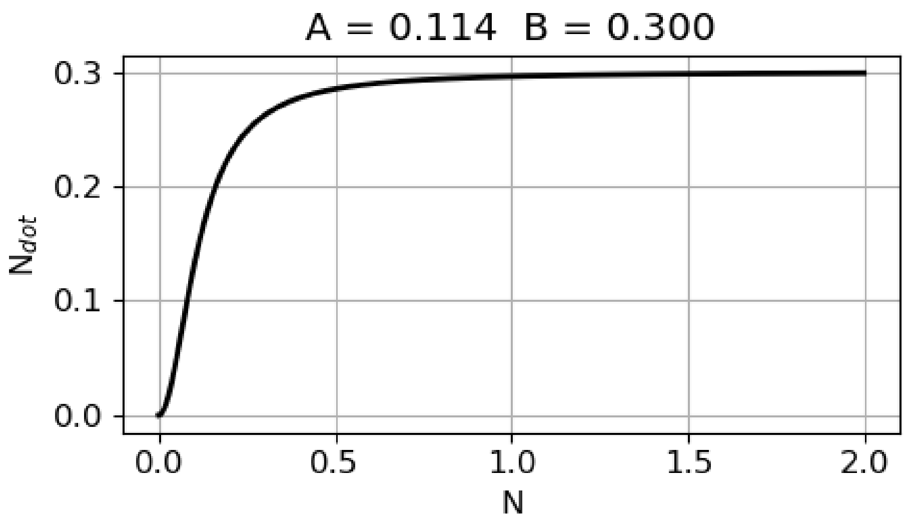

Fig. 2A. Predation curve (equation 3).

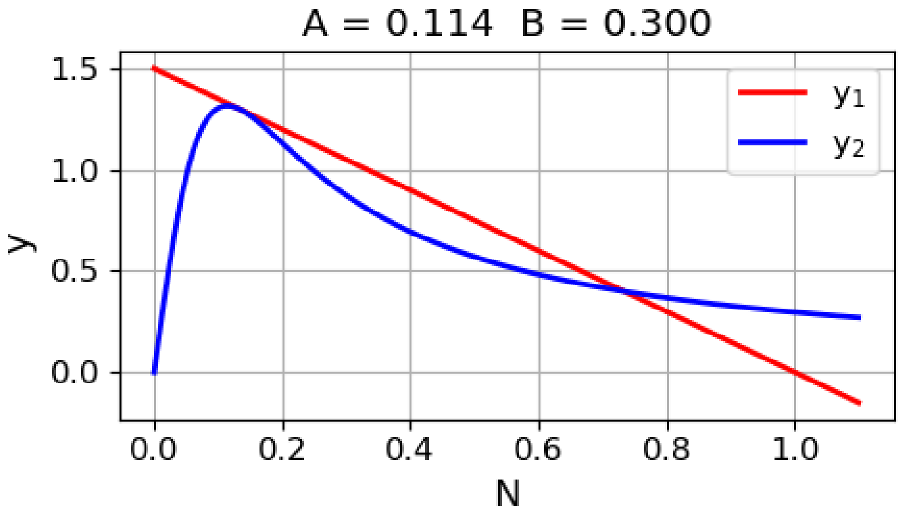

Fig. 2B. Equation 6 plots.

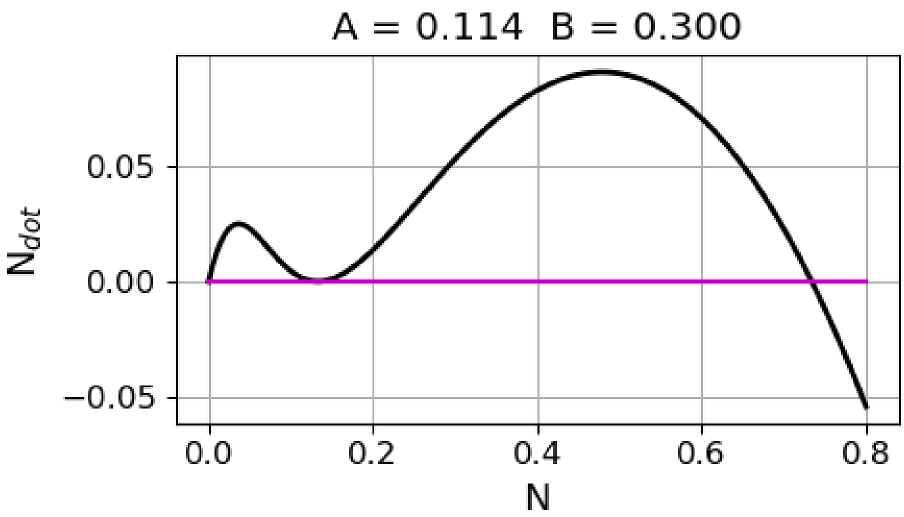

Fig. 2C. Equation 4 plot. There are four fixed points for A = 0.100 and B = 0.300.

$x_e = 0$ **unstable** (positive slope)

$x_e \sim 0.13$ semi-stable (negative / positive slope)

$x_e = 0.7335$ **stable** (negative slope)

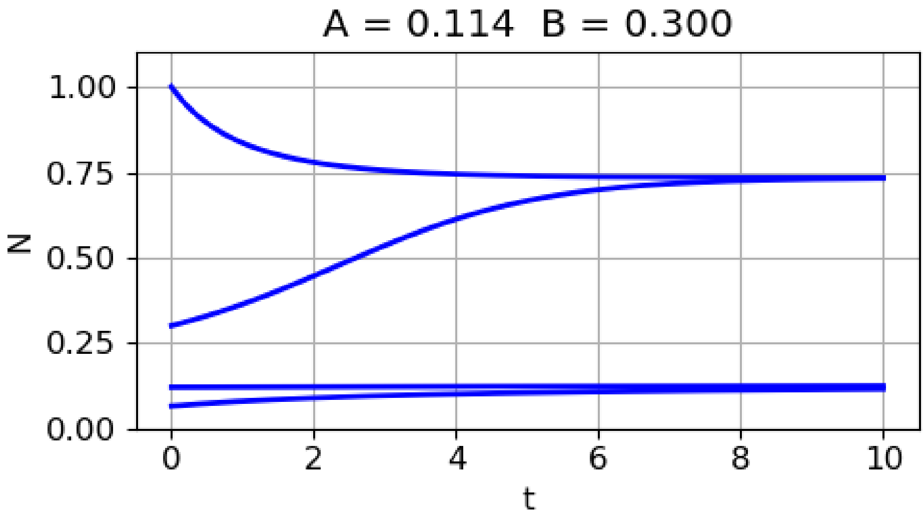

Fig 2D. Population trajectories. For different initial conditions, all trajectories converge to a stable fixed point.

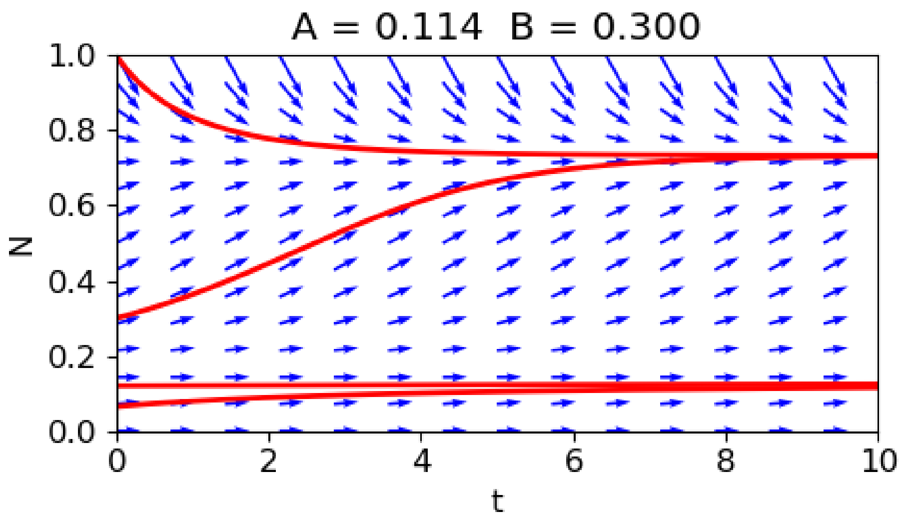

Fig. 2E. Vector field quiver plot and insect population trajectories. The trajectories are repelled from an unstable fixed point to a stable fixed point.

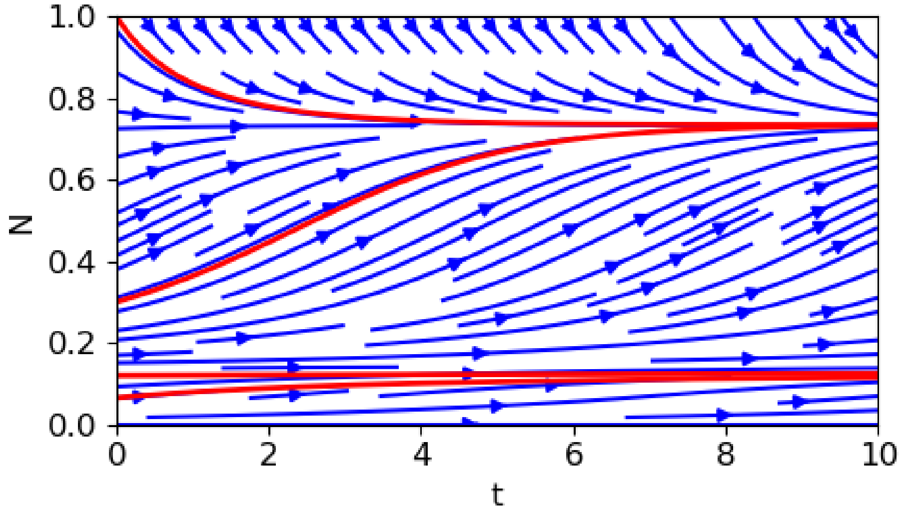

Fig. 2F. Vector field streamplot plot and insect population trajectories. The “flow” is away from an unstable fixed point and towards a stable fixed point. a stable fixed point.

Simulation 3 $r = 1.5$ $K = 1.0$ $\mathbf{A = 0.118}$ $B = 0.300$

2 fixed points

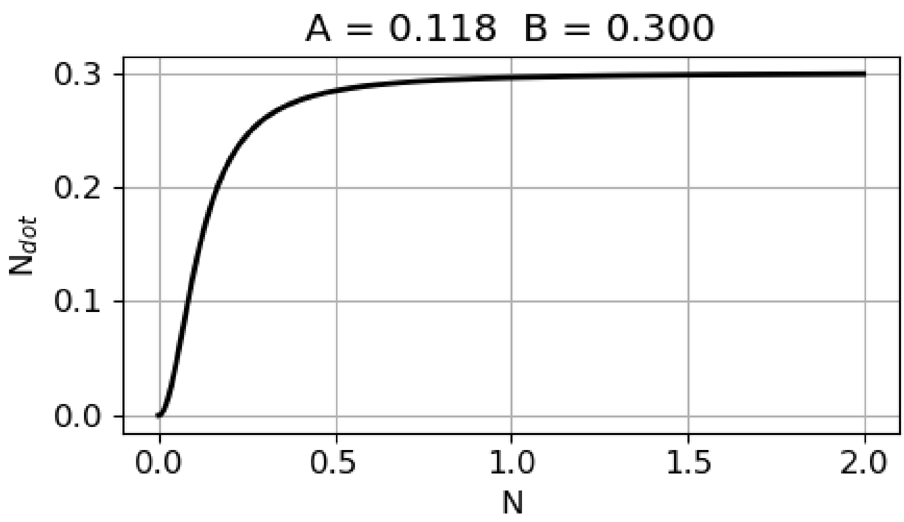

Fig. 3A. Predation curve (equation 3).

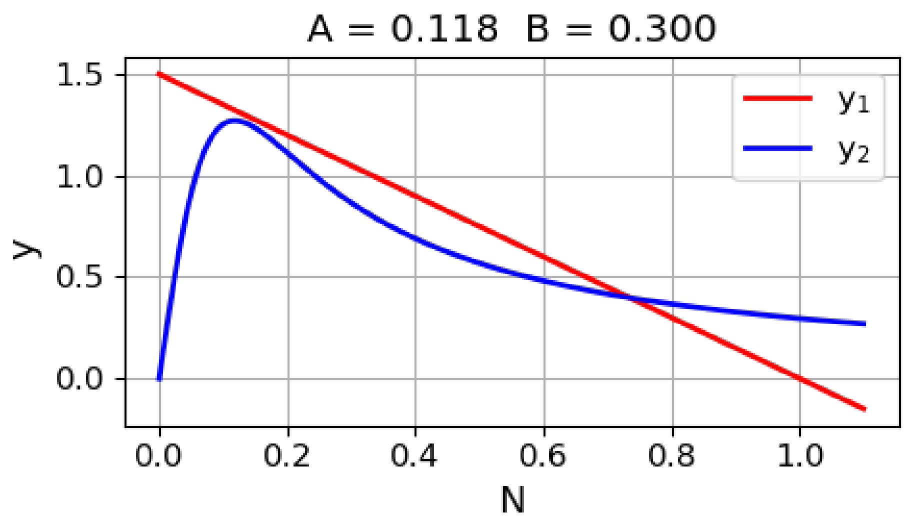

Fig. 3B. Equation 6 plots.

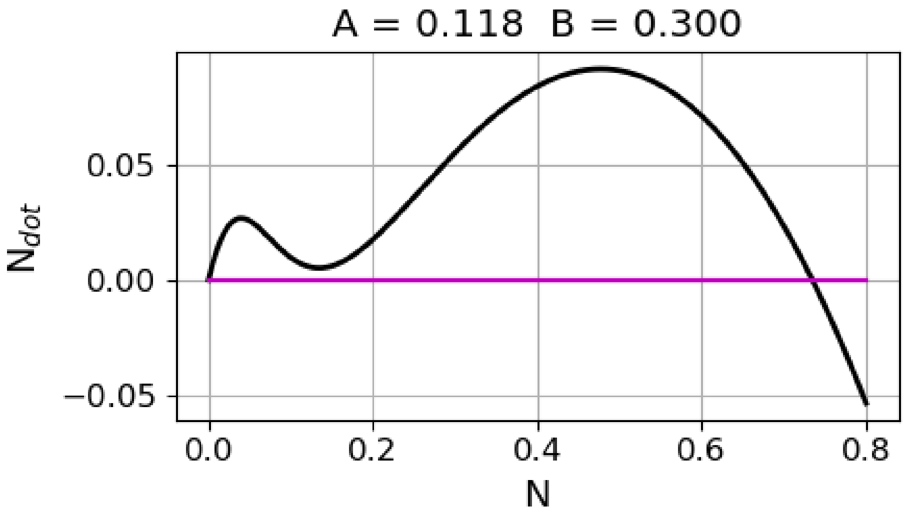

Fig. 3C. Equation 4 plot. There are four fixed points for A = 0.100 and B = 0.300.

x_e = 0 **unstable** (positive slope)

x_e = 0.7343 stable (negative slope)

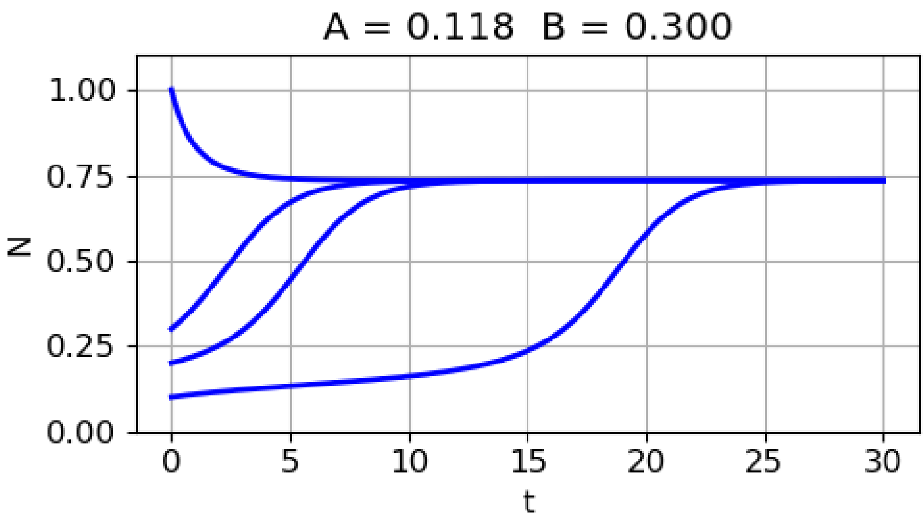

Fig 3D. Population trajectories. For different initial conditions, all trajectories converge to a stable fixed point.

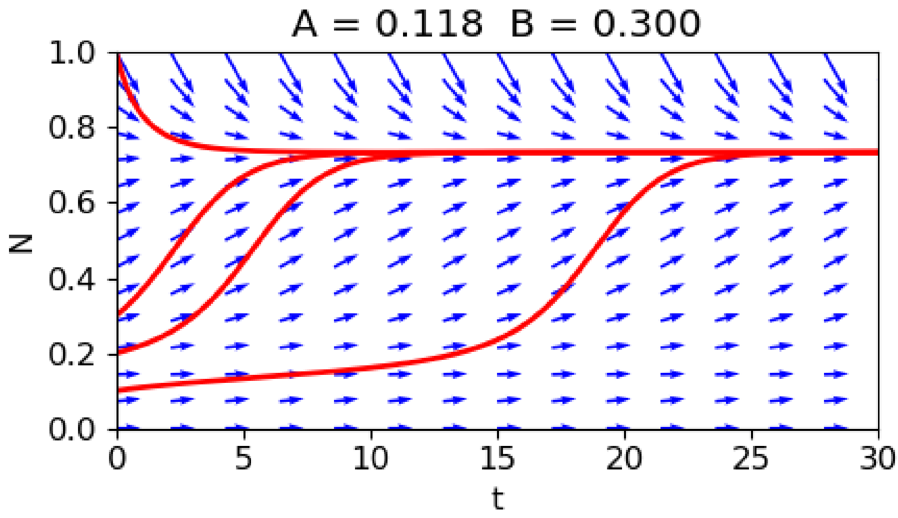

Fig. 3E. Vector field quiver plot and insect population trajectories. The trajectories are repelled from an unstable fixed point to a stable fixed point.

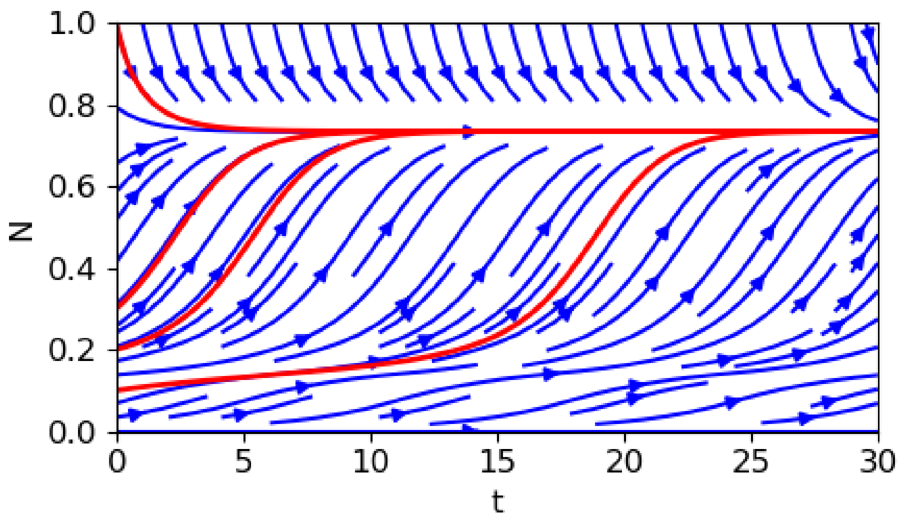

Fig. 3F. Vector field streamplot plot and insect population trajectories. The "flow" is away from an unstable fixed point and towards a stable fixed point. a stable fixed point.

The critical value of the bifurcation parameter A is

$$A_C \approx 0.114$$
$$A < A_C \Rightarrow 4 \text{ fixed points}$$
$$A = A_C \Rightarrow 3 \text{ fixed points}$$
$$A > A_C \Rightarrow 2 \text{ fixed points}$$

Not counting the fixed point $N_e = 0$ as A is varied there are two saddle node bifurcation.