# DOING PHYSICS WITH PYTHON

## DYNAMICAL SYSTEMS [1D]

## FLOW IN A CIRCLE

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## DOWNLOAD DIRECTORIES FOR PYTHON CODE

**Google drive**

**GitHub**

**ds25L11.py**

You will need to make small modifications to the Code for different examples.

**Jason Bramburger**

Flows on the Circle - Dynamical Systems | Lecture 11

https://www.youtube.com/watch?v=jcFSI7tn8tY

## INTRODUCTION

We are going to consider through a number of examples. the flow along a circle where $x$ is the angle [rad] and is $\dot{x}$ the angular velocity [rad.s$^{-1}$]. The ODEs for the system are of the form

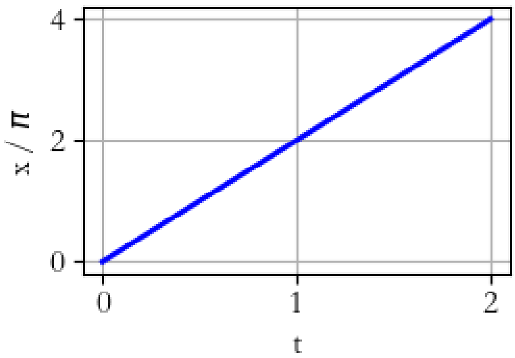
$$\dot{x} = f\left(x\right) = \omega - a\sin(x) \quad \text{and} \quad \dot{x} = x$$

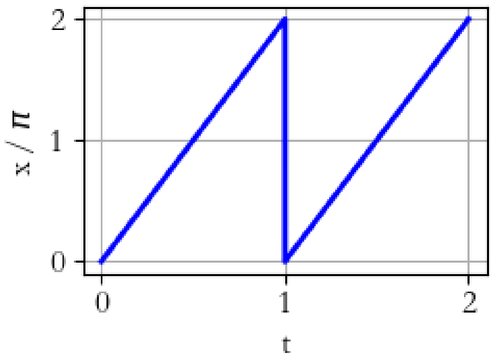
**Example 1** $\omega = 2\pi \quad a = 0 \quad \dot{x} = \omega$ rad.min$^{-1}$

This corresponds to uniform oscillations with period

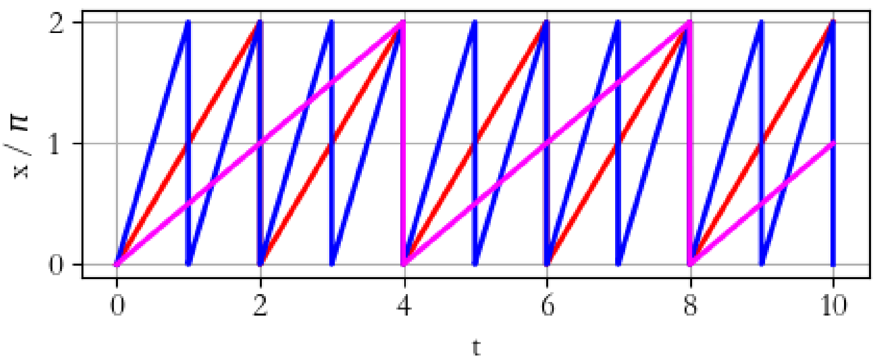
$T = 2\pi / \omega = 1.00$ min

The angle $x$ continually increases at a constant rate. There are no fixed points for this system. From any initial condition $x(0)$, the angle $x$ will just increase.




Consider three people running around a circular oval with angular velocities $\dot{x} = 2\pi, \pi, \pi / 2$ rad.min$^{-1}$. When do they overlap each other? The periods are $T$ = (**1**, **2**, **4**) min. We can this question graphically by plotting $x$ against $t$.




Blue and red runners overlap every 2.00 min, blue and magenta runners overlap every 1.33 min and the red and magenta runners overlap every 4.00 min.

We can also determine the overlap times by using the fact that the phase difference between two runners passing each other is $2\pi$.

$$\phi = 2\pi = |x_2 - x_1| \qquad \dot{\phi} = |\dot{x}_2 - \dot{x}_1| = |\omega_2 - \omega_1|$$

$$\phi = 2\pi = |(\omega_2 - \omega_1)|T = \left|\frac{2\pi}{T_2} - \frac{2\pi}{T_1}\right| T$$

$$T = \frac{1}{|1/T_2 - 1/T_1|}$$

**Example 2** $\quad \omega = 0 \quad a = -1$

$$\dot{x} = \sin(x) \qquad \text{rad.s}^{-1}$$

There are two fixed points $x_{ss}$ for the system

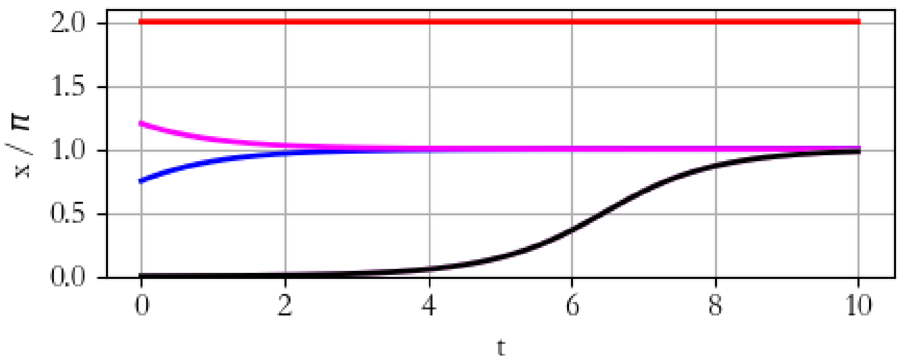
$$\dot{x} = 0 \quad \sin(x_{ss}) = 0 \quad \Rightarrow x_{ss} = 0 \quad x_{ss} = \pi$$

Stability of the two fixed points

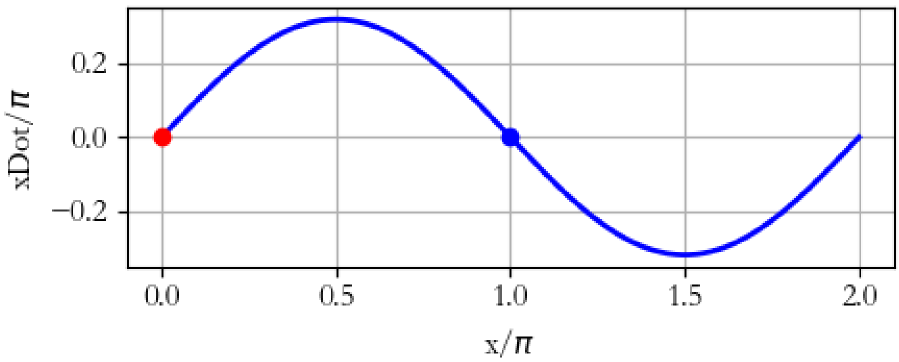
$$f(x) = \sin(x) \qquad f'(x) = \cos(x)$$

$$f'(0) = \cos(0) = 1 > 0 \Rightarrow \text{ unstable}$$

$$f'(\pi) = \cos(\pi) = -1 < 0 \Rightarrow \text{ stable}$$



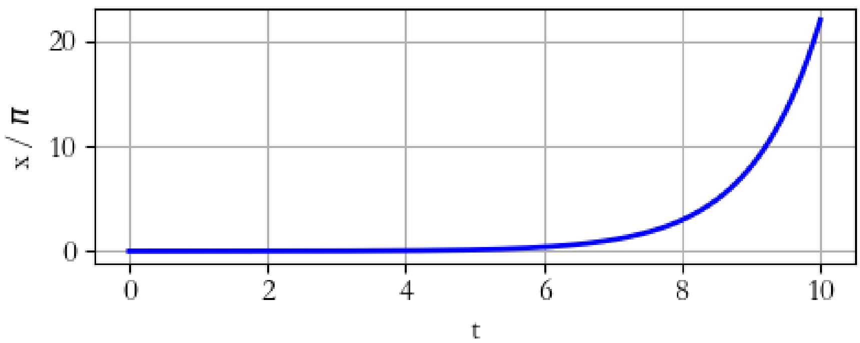

For all trajectories where $x(0) \neq x_{ss}$ converge to the stable fixed point $x_{ss} = \pi$.

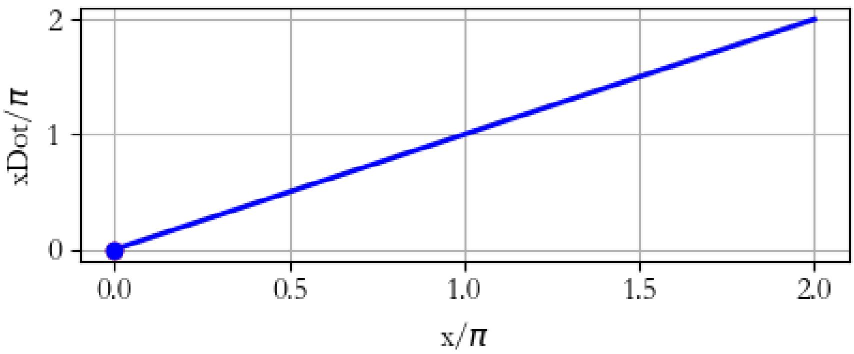



Fixed point $x_{ss} = 0$ is unstable (positive slope) and fixed point $x_{ss} = \pi$ is stable (negative slope).

**Example 3** $\dot{x} = x$ Single fixed point $x_{ss} = 0$ which is unstable.

The function $\dot{x} = x$ cannot represent flow in a circle since there is no unique velocity $x = 0 \;\; \dot{x} = 0 \quad x = 2\pi \;\; \dot{x} = 2\pi$. For flow on a circle the function $f(x)$ must be periodic in $2\pi$: $f(x) = f(x + 2\pi)$



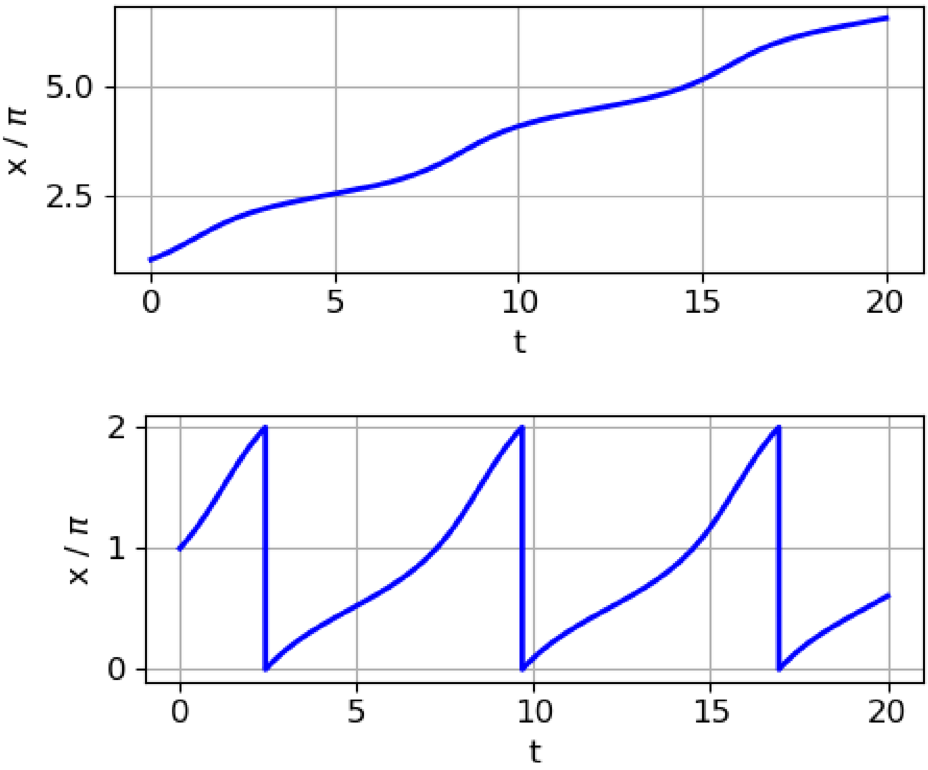




Angle $x$ increases with increasing angular velocity $\dot{x}$

**Example 4** $\dot{x} = \omega - a\sin(x)$

**4.1** $\omega > a$ $\omega = 1.0 \quad a = 0.5$

No fixed points since $\dot{x} > 0 \quad \left(\dot{x} = 0 \Rightarrow \sin(x) = \omega / a > 1\right)$.

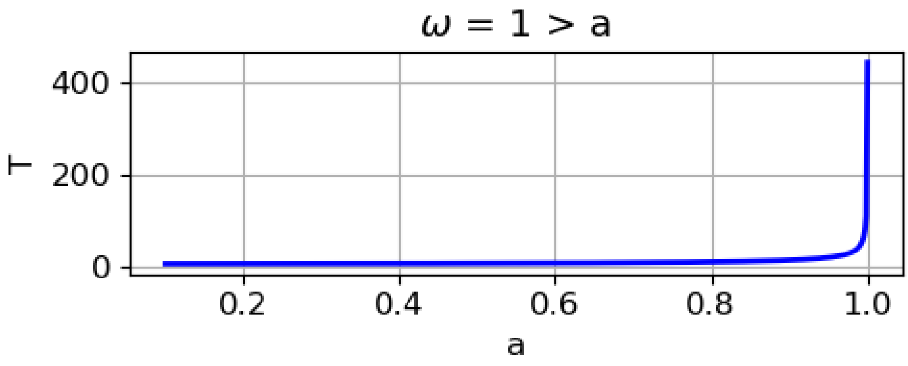



$\omega > a \qquad \omega = 1.0 \quad a = 0.5 \quad T = 7.26$

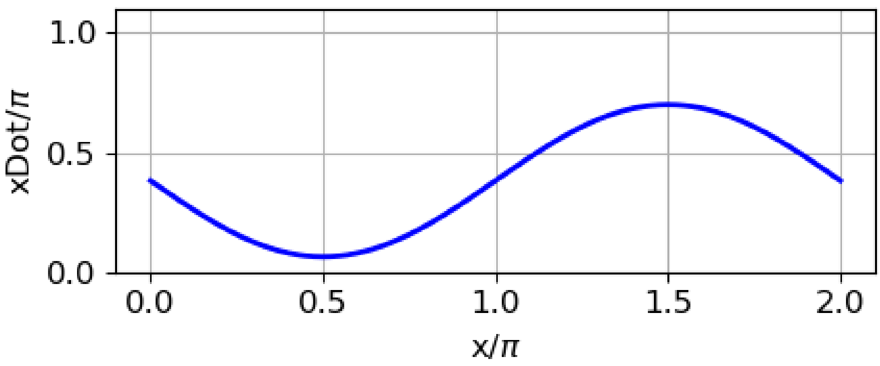
Period of oscillation $T$

$$\omega > a \quad T = \int_0^T dt = \int_0^{2\pi} \frac{dt}{dx} dx = \int_0^{2\pi} \left( \frac{1}{\omega - a\sin(x)} \right) dx$$

The integral is evaluated numerical using the Python function **simpson**.




As $a \to \omega \qquad T \to \infty \quad \Rightarrow$ *bottleneck*



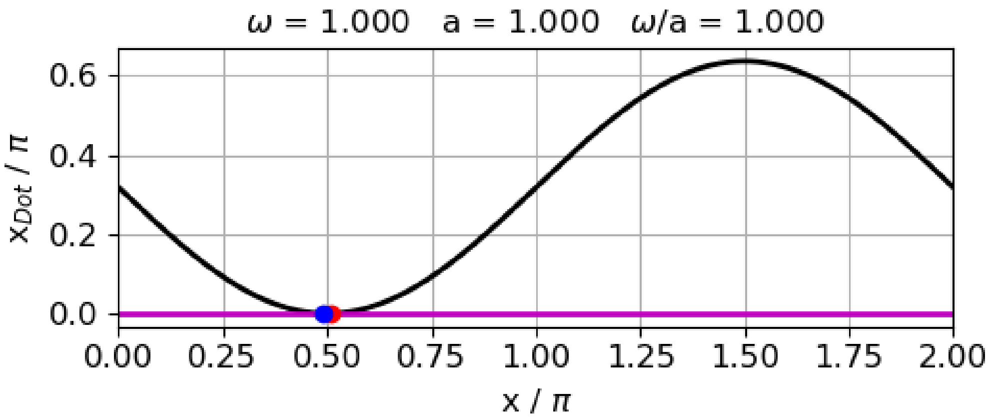

There are no fixed points as $\dot{x} > 0$. Flow is slowest at $x = \pi / 2$ and faster at $x = 3\pi / 2$. So, near $x = \pi / 2$ the flow is very slow (like a bottleneck) and rapid near $x = 3\pi / 2$. This type of behaviour can be used to model a spiking neuron.

## 4.2 $\omega = a$

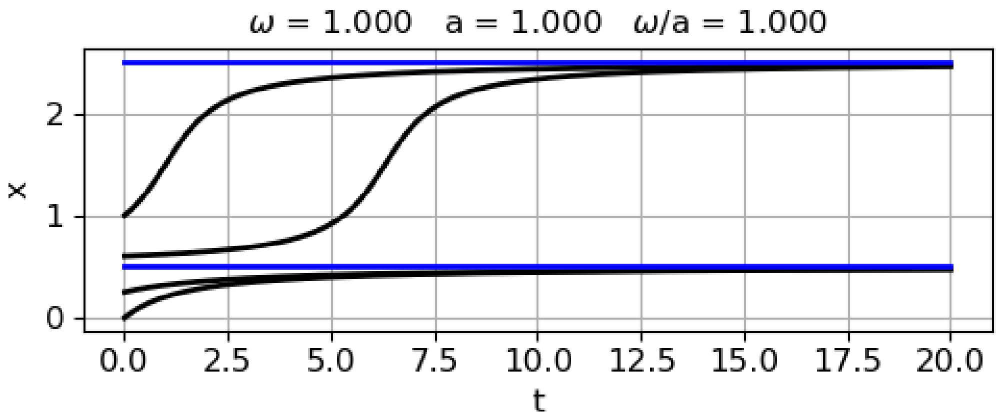
$$\omega = a \quad \dot{x} = a\left(1 - \sin(x)\right)$$

$$\dot{x} = 0 \Rightarrow \sin(x) = 1 \Rightarrow x_{ss} = \pi / 2$$

When $\omega = a$ a single fixed point emerges where $x_{ss} = \pi / 2$.




$x_{ss} = \pi / 2$ is a marginally stable fixed point




Trajectories for 4 initial conditions. The fixed point is $x_{ss} = 2.5\pi \equiv \pi / 2$ . The flow is always in a positive sense to the fixed point $x_{ss} = \pi / 2$ irrespective of the initial condition $x(0)$. The fixed point $x_{ss} = \pi / 2$ is **marginally stable**.

## 4.3 $\omega < a$

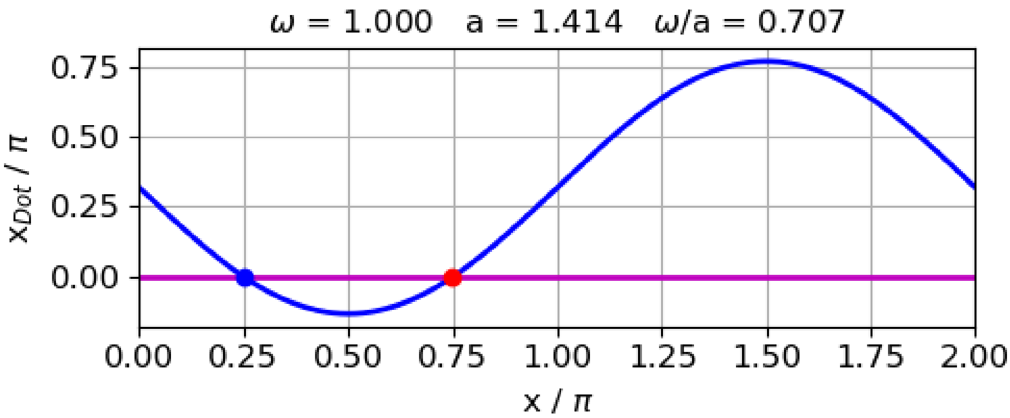
$$\omega < a \quad \dot{x} = 0 \Rightarrow \omega - a\sin(x) = 0 \quad \Rightarrow \quad \sin\left(x_{ss}\right) = \frac{\omega}{a} < 1$$

$$f(x) = \omega - a\sin(x) \quad f'(x) = -a\cos(x)$$

$$\omega = 1 \quad a = \sqrt{2} \quad \omega / a = 1/\sqrt{2} < 1$$

$$x_{ss} = \pi / 4 \quad f'(\pi / 4) = -\sqrt{2}\cos(\pi / 4) = -1 < 0 \qquad \text{stable}$$

$$x_{ss} = 3\pi / 4 \quad f'(3\pi / 4) = -\sqrt{2}\cos(3\pi / 4) = +1 > 0 \qquad \text{unstable}$$

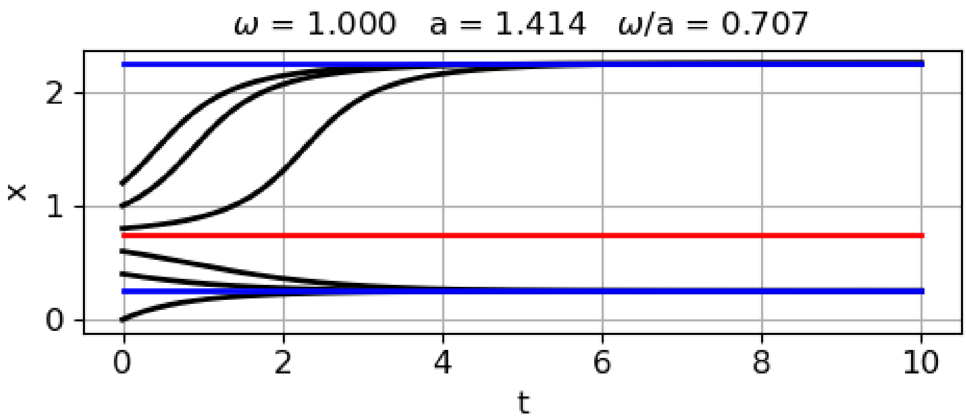



There are now two fixed point. Slope of the function $\dot{x}$ is negative at $x_{ss} = (1/4)\pi$ implies stable fixed point and the positive slope at $x_{ss} = (3/4)\pi$ implies an unstable fixed point.

$$0 < x(0) < 0.25\pi \qquad \Rightarrow \text{flow positive (anticlockwise)}$$

$$0.25\pi < x(0) < 0.75\pi \quad \Rightarrow \text{flow negative (clockwise)}$$

$$0.75\pi < x(0) < 2\pi \qquad \Rightarrow \text{flow positive (anticlockwise)}$$




Trajectories with different initial conditions all converge to the stable fixed point $x_{ss} = 0.25\pi$ $(2.25\pi)$. The unstable fixed point is $x_{ss} = 0.75\pi$.

We get a **saddle node bifurcation**

$\omega > a$ no fixed points

$\omega = a$ one fixed point

$\omega < a$ two fixed points emerge