DOING PHYSICS WITH PYTHON

DYNAMICAL SYSTEMS [1D]

GHOSTS AND BOTTLENECKS

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DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive

GitHub

ds25L12.py **ds25L12N.py**

Jason Bramburger
Ghosts and Bottlenecks - Dynamical Systems | Lecture 12

https://www.youtube.com/watch?v=4cxhMwmVcnE

Reviews

Saddle node bifurcations

Flow on a circle

INTRODUCTION

This article investigates saddle-node bifurcations at the bifurcation point where fixed points are created. The result is a "ghost fixed point" that leads to a bottleneck in the time evolution of the system. We show that these bottlenecks should be expected near the onset of a saddle-node bifurcation, no matter what the system looks like.

Example 1 ds25L12.py

We will consider the system studied in lecture 11.

$$\dot{x} = f\left(x\right) = \omega - a\,\sin(x) \quad \text{where } \omega \geq a$$

$$\omega = a \quad \dot{x} = a\left(1 - \sin(x)\right)$$

$$\dot{x} = 0 \Rightarrow \sin(x) = 1 \Rightarrow x_{ss} = \pi / 2$$

When $\omega = a$ a single fixed point emerges where $x_{ss} = \pi / 2$.

$\omega > a$

No fixed points for all times since

$$\dot{x}(t) > 0 \qquad \dot{x} = 0 \;\Rightarrow\; \sin(x) = \omega / a > 1$$

Since the angular velocity is always positive $\dot{x}(t) > 0$, the flow will be continuous on the circle in an anticlockwise sense. The period of oscillation T is

$$T = \int_0^T dt = \int_0^{2\pi} \frac{dt}{dx}\,dx = \int_0^{2\pi} \left(\frac{1}{\omega - a\sin(x)} \right) dx$$

The integral is evaluated numerical using the Python function **simpson**.

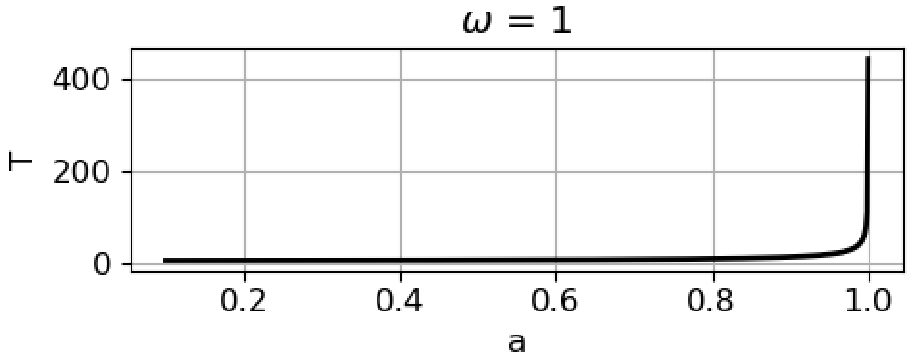

Fig. 1. The period of oscillation T as a function of the bifurcation parameter a when $\omega > a$. As $a \to \omega \Rightarrow T \to \infty$.

Figure 2 shows a series of plots of $\dot{x}$ as a function of the angle x. When $a = \omega$ there is a single fixed point at $x = \pi / 2$ and when $a > \omega$ there are two fixed points, one stable and the other unstable. For the case when $\omega > a$, the flow is slowest at $x = \pi / 2$ (minimum of $\dot{x}$). As a gets closer to ω $(a \to \omega)$ the minimum value of $\dot{x}$ approaches zero and thus the period of oscillation increases towards infinity. In this sense the angle $x = \pi / 2$ acts as a "ghost" fixed point and there is a bottleneck in the flow in the region surrounding $x = \pi / 2$ where the saddle node bifurcation occurs.

Figure 3 shows the trajectory of x as a function of time t as the value of a gets closer and closer ω. The trajectories show the bottleneck region very clearly and the "ghost" fixed point at $x = \pi / 2$.

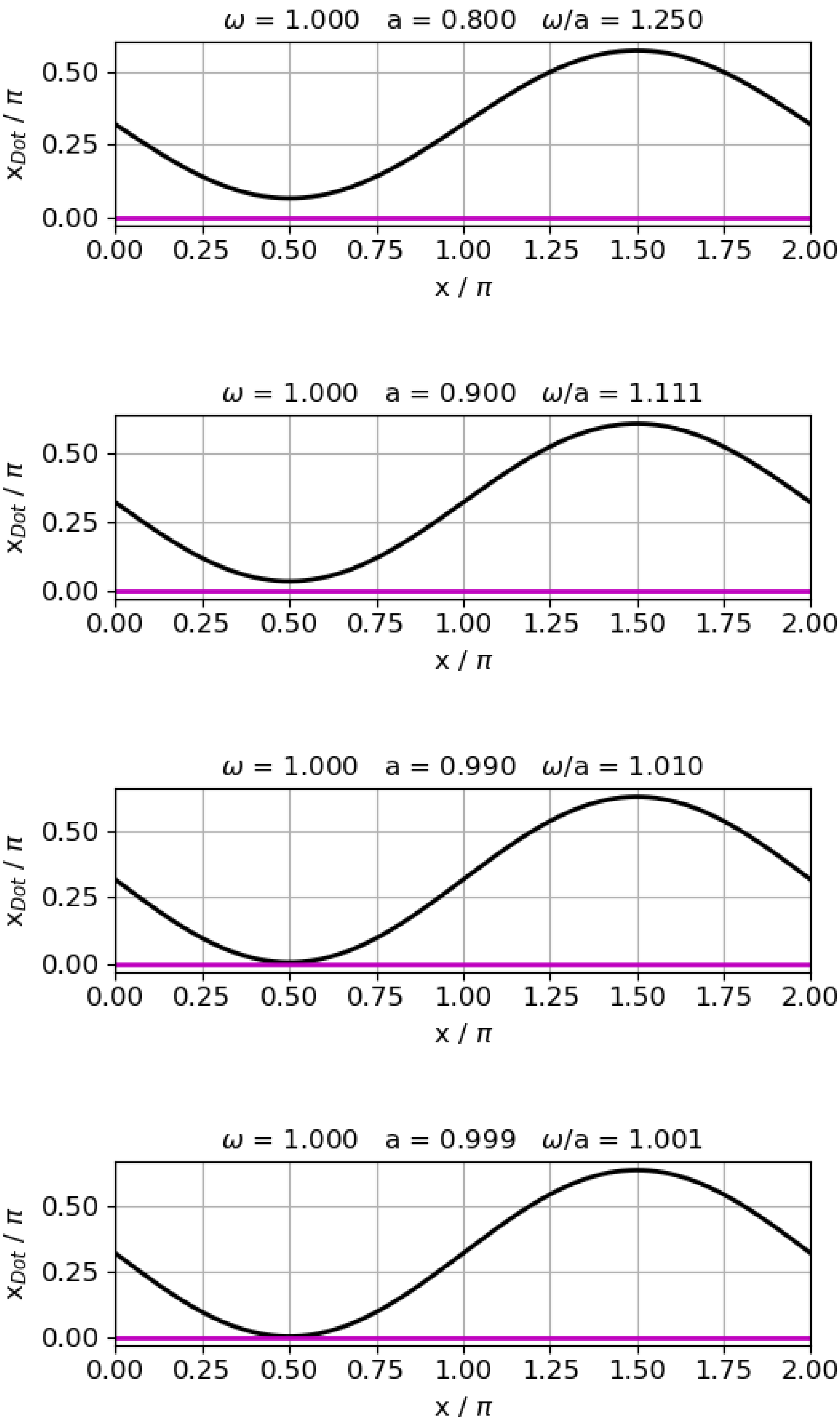

Fig. 2. $\dot{x}$ as a function of a.

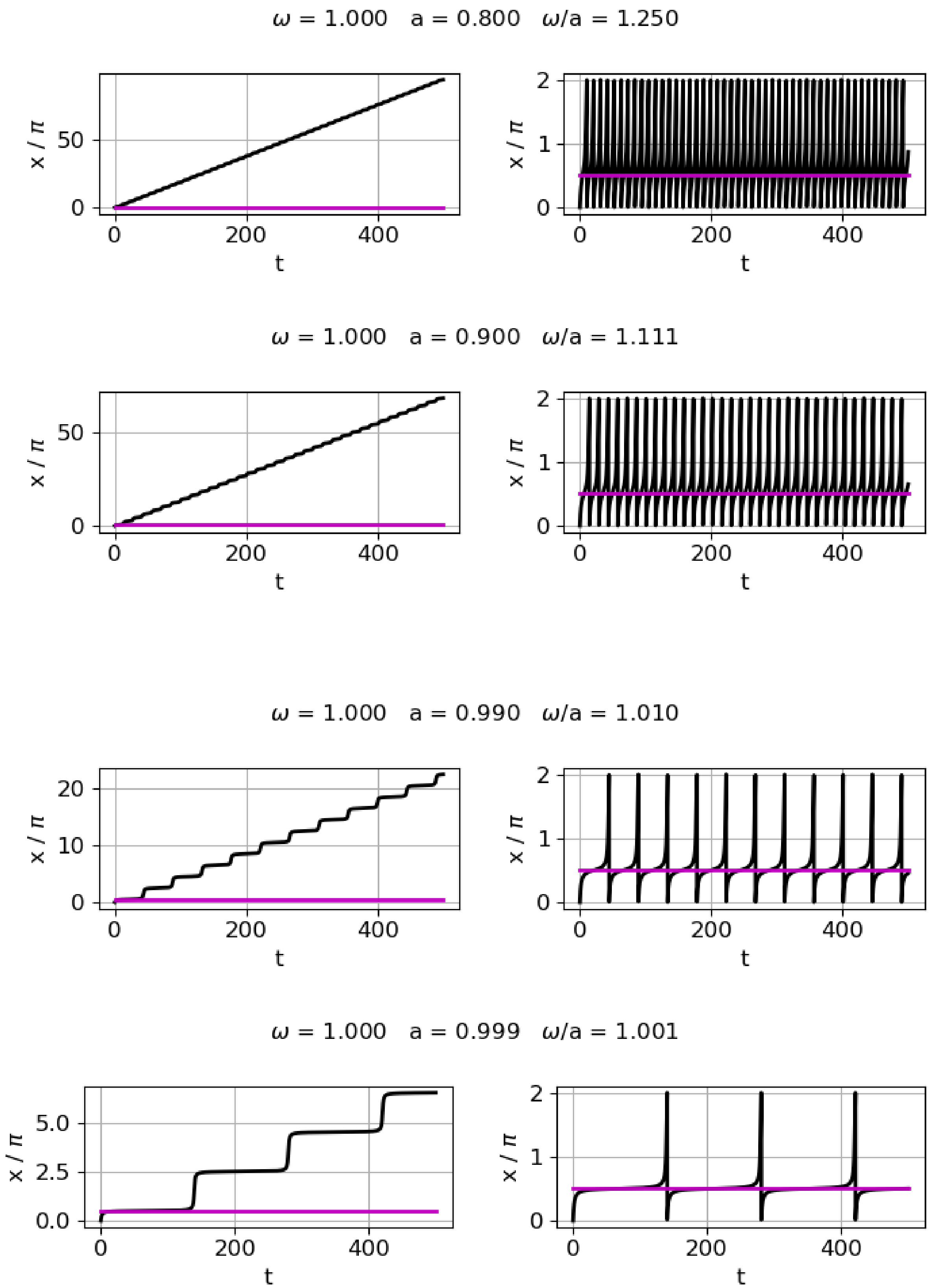

Fig. 3. Flow trajectories.

As $\omega > a \quad a \to \omega$ in the bottleneck region the flow becomes slow and the period of oscillation becomes very large. This event is a precursor to a saddle node bifurcation. At the bifurcation point two fixed points are created when $a > \omega$. This trend of the flow around the circle becoming impeded as $\omega > a \quad a \to \omega$ applies to all saddle node bifurcations.

The time spent in a bottleneck applies to all saddle node bifurcations.

We can consider the normal form of a saddle node bifurcation system. It can be shown that bifurcation systems can be approximated by the normal form.

SUBCRITICAL SADDLE NODE BIFURCATION cs100.py

The ODE for the normal form of a saddle node bifurcation is

$$\dot{x}(t) = r + x(t)^2 \qquad r \text{ is an adjustable constant}$$

$$f(x) = r + x^2 \quad f'(x) = 2x$$

$$\dot{x} = 0 \ \Rightarrow \ x_e = \pm\sqrt{-r}$$

Thus, there are three possible fixed points;

$r > 0$ no fixed points

$r = 0$ one fixed point $x_e = 0$

$r < 0$ two fixed points $x_e = -\sqrt{-r} \quad x_e = +\sqrt{-r}$

Example 2 Normal form subcritical saddle node bifurcation

ds25L12N.py

When $r > 0$ there are no fixed points and the flow is always positive and increases with time t as x increases. We can examine the flow when $r > 0$ for the normal form of a saddle node bifurcation as r decreases to zero. A bottleneck occurs for small values of r near the bifurcation point $x_{SS} = 0$ ($r = 0$). This bottleneck behaviour is clearly shown in figures 4, 5, and 6.

The time T for the flow to go from $x = 0$ to $x = x_{Max}$ is calculated from the ODE equation by integration. The integration is done numerically using the Python function **simpson**. We can thus calculate the dependence of T on the bifurcation parameter r (figure 4).

$$\dot{x}(t) = r + x(t)^2$$

$$T(r) = \int_0^{t_{Max}} dt = \int_0^{x_{Max}} \frac{dx}{r + x(t)^2}$$

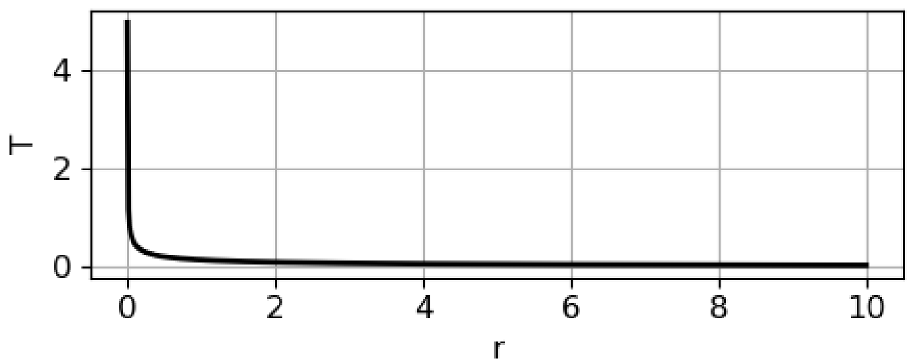

Fig. 4. A bottleneck occurs when $r \to 0$. T is the time taken for the flow from $x = 0$ to $x = 10$.

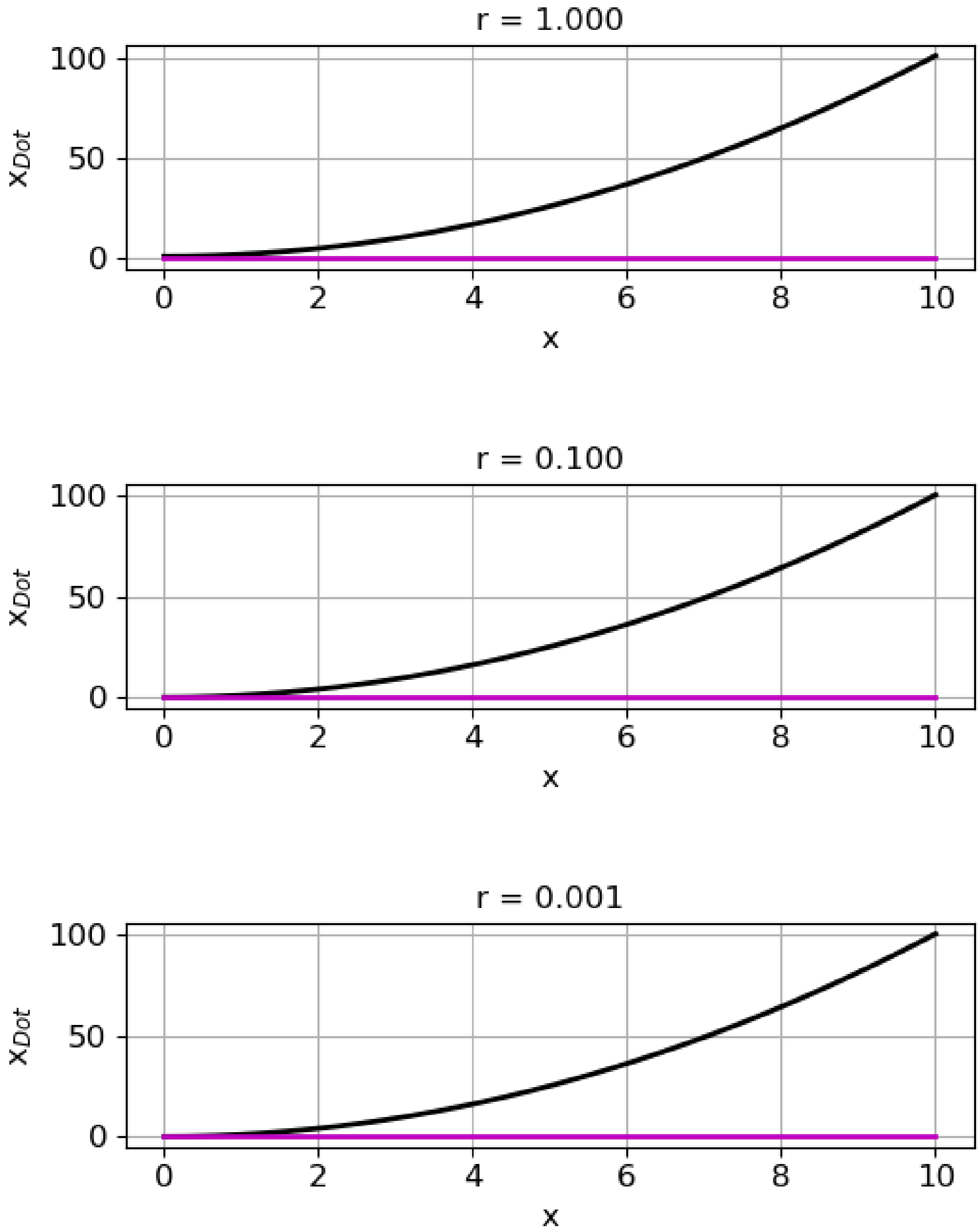

Fig. 5. As $x(t)$ increases then $\dot{x}(t)$ becomes independent of r.

$$\dot{x}(t) \approx x(t)^2$$

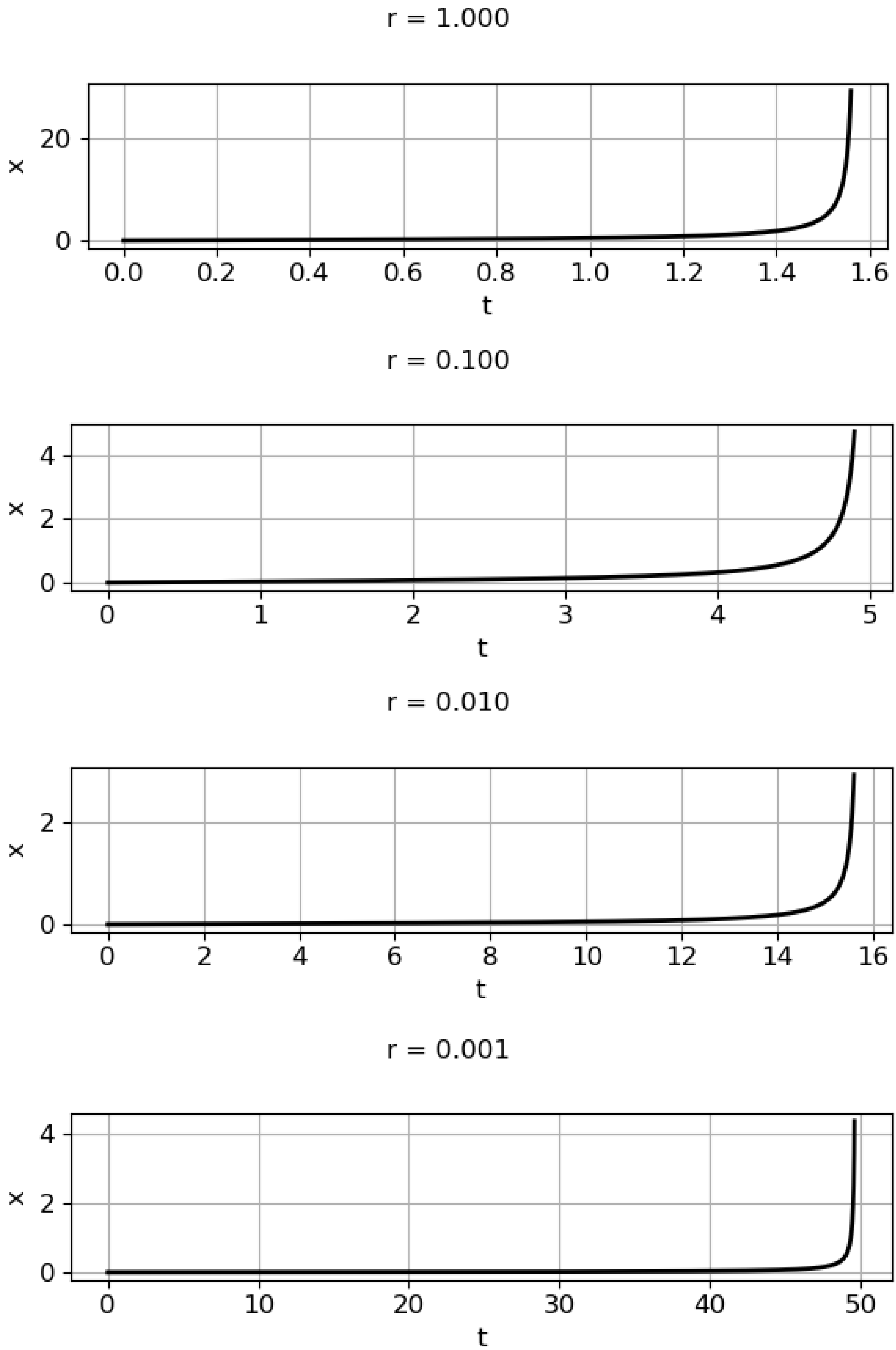

Fig. 6. As $r \to 0$ the time in the bottleneck increases dramatically.