DOING PHYSICS WITH PYTHON

DYNAMICAL SYSTEMS [1D]
FIXED POINTS AND STABILITY

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DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive

GitHub

ds25L3.py $\dot{x} = x^2 - 1$

ds25L3A.py $\dot{x} = x - \cos(x)$

ds25L3B.py Population growth $\dot{N} = r\,N\left(1 - \dfrac{N}{K}\right)$

Jason Bramburger
Fixed Points and Stability - Dynamical Systems | Lecture 3

https://www.youtube.com/watch?v=BlBbPYuQyz0

INTRODUCTION

In this article we discuss fixed points of [1D] dynamical systems Fixed points go by many different names depending on the discipline, including steady-states, equilibria, equilibrium points, and rest-states They all mean the same thing. We introduce the basics of fixed points and discuss what it means for them to be stable. We analyse stability using a number of approaches.

STABILITY

We can look at the mathematics defining the stability of fixed points. Consider the function $d(t)$ for the difference between the solution $x(t)$ and the fixed point x_{ss}

$$d(t) = x(t) - x_{ss}$$

If $d(t)$ increases with time, then x_{ss} is unstable or stable and if $d(t)$ decreases with time.

$$\dot{d}(t) = \dot{x}(t) = f(x) = f(d + x_{ss})$$

Using the Taylor expansion about x_{ss}

$$\dot{d}(t) = \dot{x}(t) = f(x) = f(d + x_{ss})$$

$$f(x) = f(x_{ss}) + f\,'(x_{ss})\,d + O(d^2)$$

$$f(x_{ss}) = 0 \quad O(d^2) \approx 0$$

$$f(x) = f\,'(x_{ss})\,d = \lambda\, d \qquad \lambda = f\,'(x_{ss})$$

$$\dot{d} = \lambda\, d$$

The solution of the ODE $\dot{d} = \lambda\, d$ gives either exponential growth or decay

$$d = d_0 e^{\lambda t} \qquad d_0 = d(0)$$

$\Rightarrow\ \lambda = f\,'(x_{ss}) > 0$ exponential growth $t \to \infty$ $x(t) \to \pm\infty$

Unstable

$\Rightarrow\ \lambda = f\,'(x_{ss}) < 0$ exponential decay $t \to \infty$ $x(t) \to x_{ss}$

Stable

SIMULATIONS

Example 1 **ds25L3.py** $\dot{x} = x^2 - 1$

$$\dot{x} = x^2 - 1 \qquad \text{initial condition } x(0) = x_0$$

This equation can be solved numerically using the Python function **odeint**.

The steady-state solutions are

$$\dot{x} = x_{ss}^{\;2} - 1 = 0 \Rightarrow x_{ss} = -1 \;\text{ and }\; x_{ss} = +1$$

where x_{ss} is a fixed-point of the system.

To determine the stability of each fixed point, let

$$f(x) = x^2 - 1 \quad f'(x) = 2x$$

then

$f'(x_{ss}) < 0$ **stable fixed point**

$\Rightarrow$ the flow is decreasing and moving to left (-x direction)

$f'(x_{ss}) > 0$ **unstable fixed point**

$\Rightarrow$ the flow is increasing and moving to right (+x direction)

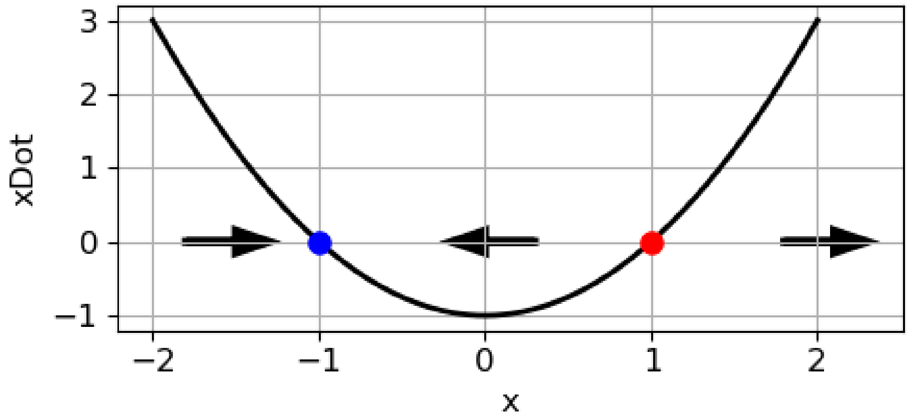
Thus,

$x_{ss} = -1 \;\Rightarrow\; f'(-1) = -2 < 0$ **stable fixed point**

$x_{ss} = +1 \;\Rightarrow\; f'(1) = 2 > 0$ **unstable fixed point**

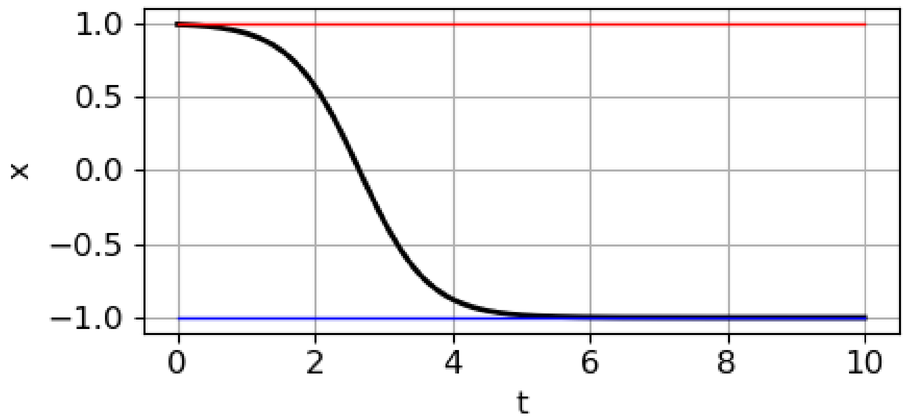
Fixed point $x_{ss} \Rightarrow f(x_{ss}) = 0 \Rightarrow x = x_{ss} \ \forall \ t$

x_{ss} is **stable** if $x(0)$ 'close' to x_{ss} then $x(t)$ will stay 'close' to x_{ss}

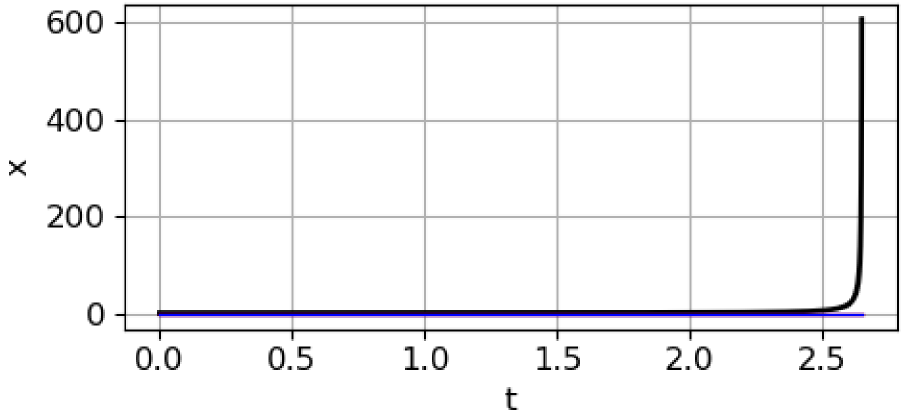


Stable fixed point $x_{ss} = -1$: the flow is pulled into $x = -1$ and the fixed point acts as a sink or an attractor.

Unstable fixed point $x_{ss} = +1$: the flow is pushed away from $x = +1$ and the fixed point acts as a source or a repeller.



$x(0) = 0.99$



$$x(0) = 1.01$$

$$x(0) < -1 \Rightarrow \; t \to \infty \;\; x \to -1$$
$$x(0) < +1 \Rightarrow \; t \to \infty \;\; x \to -1$$
$$x(0) > +1 \Rightarrow \; t \to \infty \;\; x \to +\infty$$

Example 2 **ds25L3A.py** $\dot{x} = x - \cos(x) \qquad -\pi \le x \le +\pi$

The steady-state solutions are

(3) $\dot{x} = x_{ss} - \cos(x_{ss}) = 0 \Rightarrow x_{ss} = 0.7391$

where x_{ss} is a fixed-point of the system.

The value of x_{ss} is calculated using the Python function **fsolve**

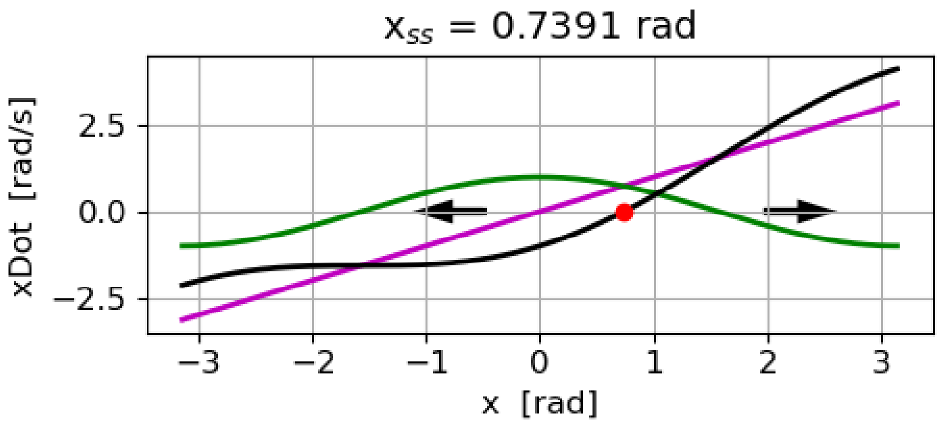
```
# fixed points
def equations(variables):
   Z = variables  # Unpack the variables
   eq = Z - cos(Z)
   return eq
IC = [1.0] # Initial guess for x and y
xss = fsolve(equations, IC)
```

To determine the stability of each fixed point, let

$$f(x) = x - \cos(x) \qquad f'(x) = 1 + \sin(x)$$

$$f'(x_{ss} = 0.7391) = 1 + \sin(0.7391) = 1.67 > 0$$

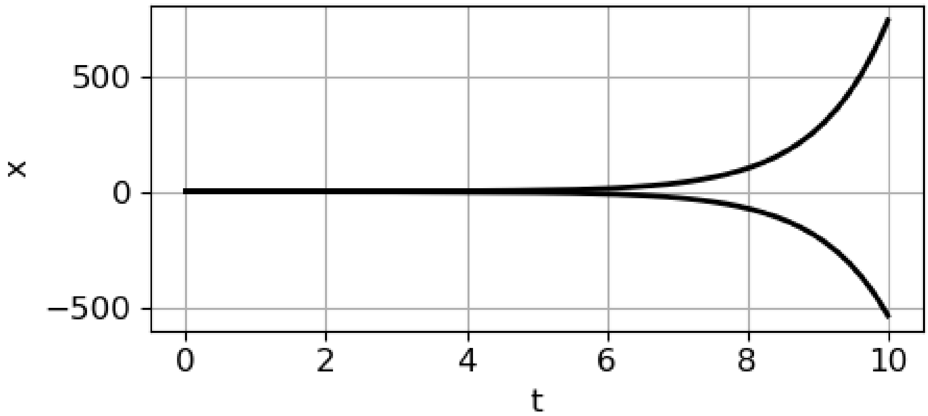
The fixed point **x_{ss} = 0.7391** is **unstable**



$$x > \cos(x) \quad f'(x) > 0 \;\Rightarrow\; \text{flow in direction } x \text{ increasing}$$

$$x V \cos(x) \quad f'(x) < 0 \;\Rightarrow\; \text{flow in direction } x \text{ decreasing}$$

$y = x$ (magenta) $\quad y = \cos(x)$ (green) $\quad y = x - \cos(x)$ (**black**)



$$x(0) \neq x_{ss} \quad x(0) < 0.7391 \;\Rightarrow t \to \infty \;\; x \to -\infty$$

$$x(0) > 0.7391 \;\Rightarrow t \to \infty \;\; x \to +\infty$$

Example 3 Population growth ds27L3B.py

We will consider the simplest population growth model

$$\dot{N} = r\,N$$

where $\dot{N}$ is the rate of change of the population, N is the population at time t and $r > 0$ is the population growth rate. The solution to this equation is that the population grows exponentially and is unbounded (population grows for ever)

$$N = N_0\,e^{r\,t} \quad t = 0, N(0) = N_0 \qquad t \to \infty \;\Rightarrow\; N \to \infty$$

This is an unrealistic model. We can add a term to the ODE to represent the competition for limited resources which will limit the maximum size of the population to its carrying capacity K (maximum population that the environment can support).

Let the per capita growth rate be $\dot{N} / N$, then

$\dot{N} / N > 0$ if $N < K$ population will increase

$\dot{N} / N < 0$ if $N > K$ population will decrease

The simplest model for competition between resources is known as the **logistic model**

$$\dot{N} = r\,N\left(1 - \frac{N}{K}\right)$$

$N(t) > 0$ otherwise there is no population.

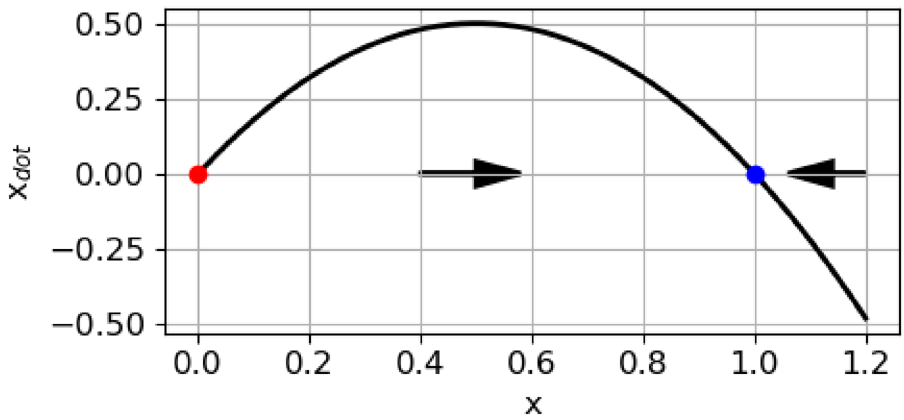
The steady-state population is given by the fixed point N_{ss}

$$\dot{N} = 0 \;\Rightarrow\; N_{ss} = K$$

Ignore $N = 0$ since it means zero population.

So, for all initial conditions $N(0) > 0$, the population will converge to the carrying capacity K.

$\boldsymbol{r = 2 \quad K = 1}$



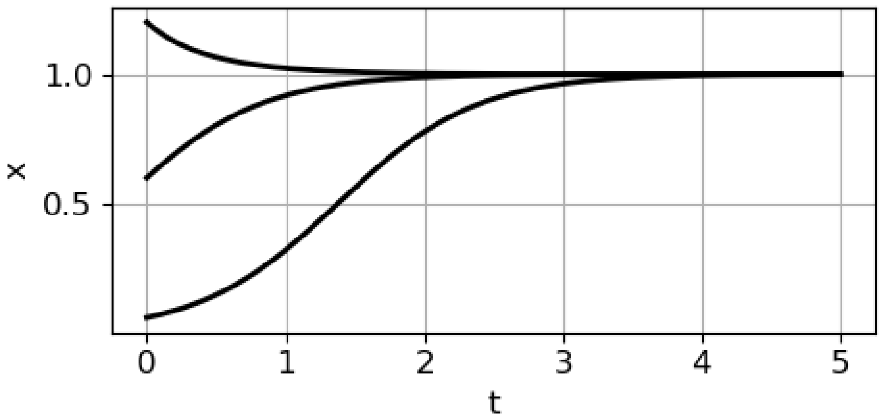
slope at $x_{ss} = 0$ is positive $\Rightarrow$ unstable

slope at $x_{ss} = 1$ is negative $\Rightarrow$ stable

$$\dot{N} = r\,N\left(1 - \frac{N}{K}\right) \qquad f'(N) = r - \frac{2N}{K} \qquad r > 0$$

$$f'(0) = r > 0 \qquad \text{unstable}$$

$$f'(K) = -r < 0 \qquad \text{stable}$$



$$N(0) \neq 0 \;\Rightarrow\; N(t) \to K$$