DOING PHYSICS WITH PYTHON

DYNAMICAL SYSTEMS [1D]

Existence and Uniqueness

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DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive

GitHub

ds25L4.py $\dot{x} = x^2 - 1$

Jason Bramburger

Existence and Uniqueness - Dynamical Systems | Lecture 4

https://www.youtube.com/watch?v=xTi_DCBjGeg

INTRODUCTION

In dynamical systems, the existence and uniqueness theorem guarantees that under specific conditions, a differential equation has a solution and that is unique. This is crucial for ensuring that a model of a system behaves predictably and that predictions made based on the model are reliable. Initial value problems (IVP) consist of a differential equation and an initial condition. The existence theorem ensures that a solution to the IVP exists, and the uniqueness guarantees that the solution is unique, meaning there are no other possible solutions that satisfy both the differential equation and the initial condition.

The theorem typically applies to first-order differential equations of the form $dx/dt = f(x, t)$. For a unique solution to exist, the function $f(x, t)$ and its partial derivative with respect to x, $\partial f/\partial x$ must be continuous at the initial condition, $x(0) = x_0$.

Often, the theorem guarantees existence and uniqueness only within a local neighbourhood of the initial condition. Extending this to a global result requires stronger conditions.

The existence and uniqueness theorem implies that only fixed points are significant, since there cannot be any oscillations. Since a solution is unique, you cannot go forward and backwards along the phase line, that is, the flow can not be such that you come back to where you start.

NO PERIODIC SOLUTION FOR $\dot{x} = f(x)$

SIMULATIONS

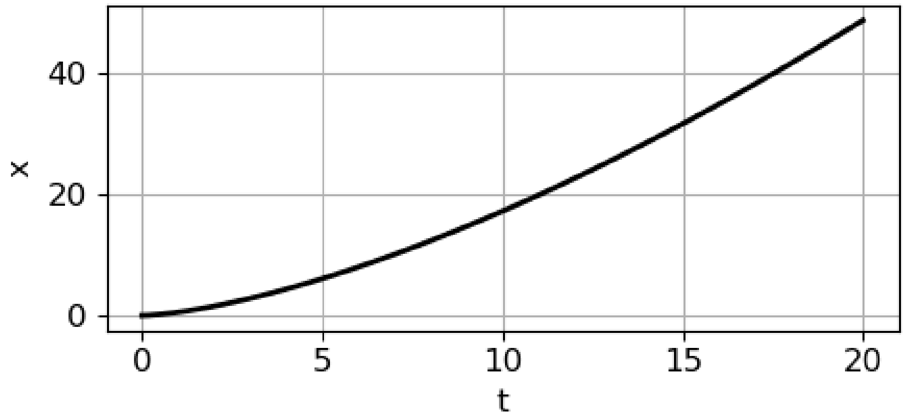
Example 1 ds25L4.py $\dot{x} = x^{1/3}$

$\dot{x} = x^{1/3}$ initial condition $x(0) = x_0 = 0$

The steady-state is obviously $x_{ss} = 0$

It is easy to show the solution to this ODE is

$$x = \left(\tfrac{2}{3}t\right)^{3/2}$$



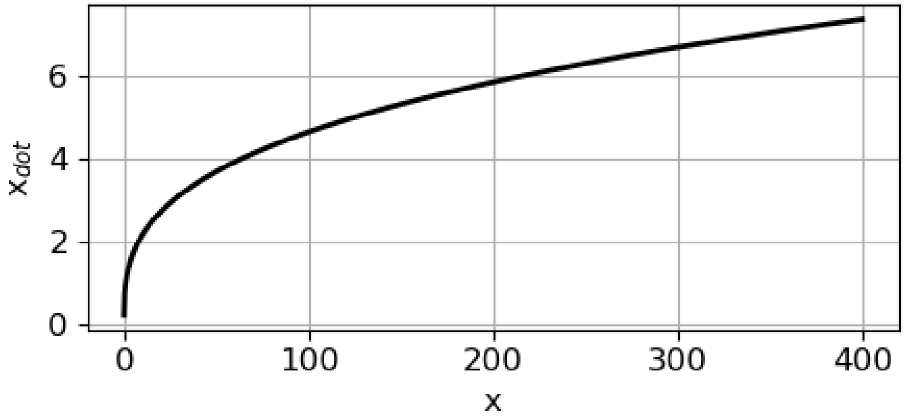
The solution grows with time from $t = 0$. But we have just said that $x = 0$ is a fixed point, and if $x_0 = 0$ then $x(t) = 0$ for all time.

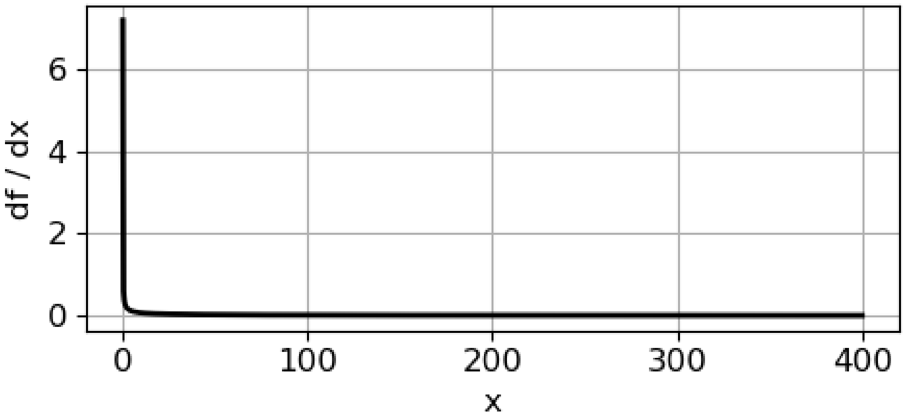
Something is wrong. There is no unique solution since the derivative of the function $f(x)$ w.r.t x is not continuous at the fixed point $x_{ss} = 0$.

$$f(x) = x^{1/3} \quad df / dx = \frac{1}{3x^{2/3}}$$

$$x_{ss} = 0 \;\Rightarrow\; df / dx\big|_{x_{ss}=0} = \infty$$

For a unique solution to exist, the function $f(x, t)$ and its partial derivative with respect to x, $\partial f/\partial x$ must be continuous at the initial condition, $x(0) = x_0$. This condition is not satisfied.





The derivative of the function $f(x) = x^{1/3}$ goes to $+\infty$ as $x \to 0$.

$$f'(x) = \frac{1}{3x^{2/3}} \quad x \to 0 \;\; f'(x) \to \infty$$

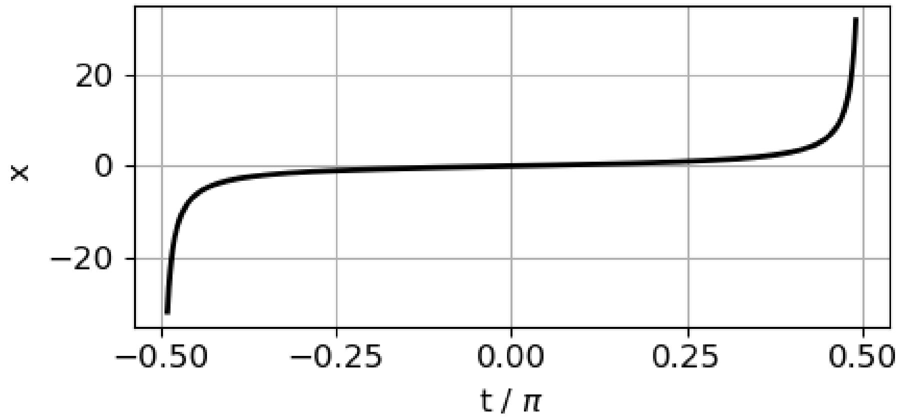
Example 2 ds25L4.py $\dot{x} = 1 + x^2$

$\dot{x} = 1 + x^2 \quad \dot{x} = 0 \Rightarrow x^2 = -1$ there is no steady-state solution.

For all x values $\dot{x} > 0 \Rightarrow$ flow to the right increases as x increases.

We can solve the ODE using Python's symbolic commands with the initial condition $x(0) = 0$

```
from sympy import symbols, integrate
x = symbols('x')
eq = 1/(1+x**2)
integral = integrate(eq, x)        → atan(x)
t = linspace(-0.98*pi/2,0.98*pi/2,N)
xt = tan(t)
```



The solution only exists in the local region $-\pi / 2 < t < \pi / 2$ since the solution "blows-up" at $t = -\pi / 2 \;\; x = -\infty$ and $t = \pi / 2 \;\; x = +\infty$. This is called a finite time blow up. If $x(0) \neq 0$ then the constant of integration changes and finite time interval for the solution will also change. This is not obvious just by looking at the equation and a phase line diagram that these problems arise. So, you must always be careful. We conclude that finite solutions may only exist for a finite time.