DOING PHYSICS WITH PYTHON

DYNAMICAL SYSTEMS [1D]

Saddle Node Bifurcations

Ian Cooper
matlabvisualphysics@gmail.com

DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive

GitHub

cs100.py (subcritical) **cs100super.py** (supercritical)
cs101.py

Jason Bramburger
Saddle Node Bifurcations - Dynamical Systems | Lecture 6
https://www.youtube.com/watch?v=gjplusyelq0

INTRODUCTION

In this article, we will dive into bifurcations of one-dimensional dynamical systems. Bifurcation in dynamical systems refers to a qualitative change in the system's behaviour as a parameter is varied. Essentially, it's a point where a small change in a parameter can cause a significant shift in how the system evolves over time, such as a

change in the number or stability of equilibrium points or the emergence of oscillations.
We will start with one of the simplest: the saddle-node bifurcation. Through examples we demonstrate that a saddle-node bifurcation arises through the manipulation of a system parameter that creates or destroys two fixed points.

A saddle-node bifurcation occurs as the bifurcation parameter is varied and two equilibrium points (one stable and one unstable) approach each other. At the bifurcation point, they collide and annihilate each other, meaning they no longer exist as solutions to the system's equations. The pair of equilibria produced in a saddle-node bifurcation always consists of one stable and one unstable equilibrium. The stable equilibrium absorbs small disturbances, while the unstable one amplifies them.

Examples

Population Dynamics: A population model might exhibit a bifurcation where a stable population level becomes unstable, and the population either crashes or explodes depending on environmental conditions.

Electrical Circuits: A circuit might have a stable state until a certain voltage is reached, at which point it might switch to an oscillating state.

Biological Systems: Cell cycle transitions, neural networks, and gene regulation can all be modelled using dynamical systems with bifurcations.

The [1D] nonlinear system's ODE can be expressed as

$$\dot{x}(t) = f\left(x(t), r\right)$$

and the fixed points of the system are

$$f\left(x_e(t), r\right) = 0$$

where r is the bifurcation parameter. So, the fixed points x_e and their stability depends upon the bifurcation parameter.

Example 1 SUBCRITICAL SADDLE NODE BIFURCATION

cs100.py

$\dot{x}(t) = r + x(t)^2$ r is an adjustable constant

$$f(x) = r + x^2 \quad f'(x) = 2x$$

$$\dot{x} = 0 \;\Rightarrow\; x_e = \pm\sqrt{-r}$$

Thus, there are three possible fixed points;

$r > 0$ no fixed points

$r = 0$ one fixed point $x_e = 0$

$r < 0$ two fixed points $x_e = -\sqrt{-r}$ $x_e = +\sqrt{-r}$

The system's behaviour can be considered in terms of the **velocity vector field**. The system vector field is represented by a vector for the velocity at each position *x*. The arrow for the velocity vector at point *x* is to the right (+X direction) if $\dot{x} > 0$ and to the left (-X direction) if $\dot{x} < 0$. So, the flow is to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$. At the points where $\dot{x} = 0$, there are no flows and such points are called **fixed points**.

***r* > 0** there are no fixed-points

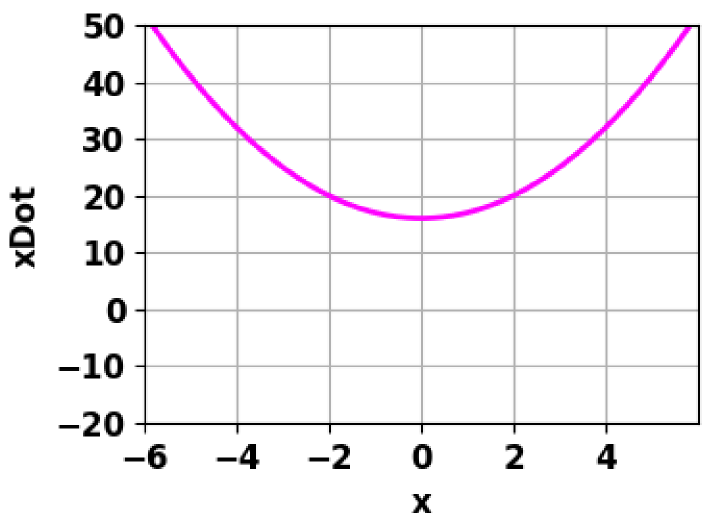


Fig. 1.1 If $r > 0$ then there are no fixed points

$$\dot{x} = r + x^2 \quad r > 0 \Rightarrow \dot{x} > 0 \text{ flow to the right } t \to \infty \; x \to +\infty$$

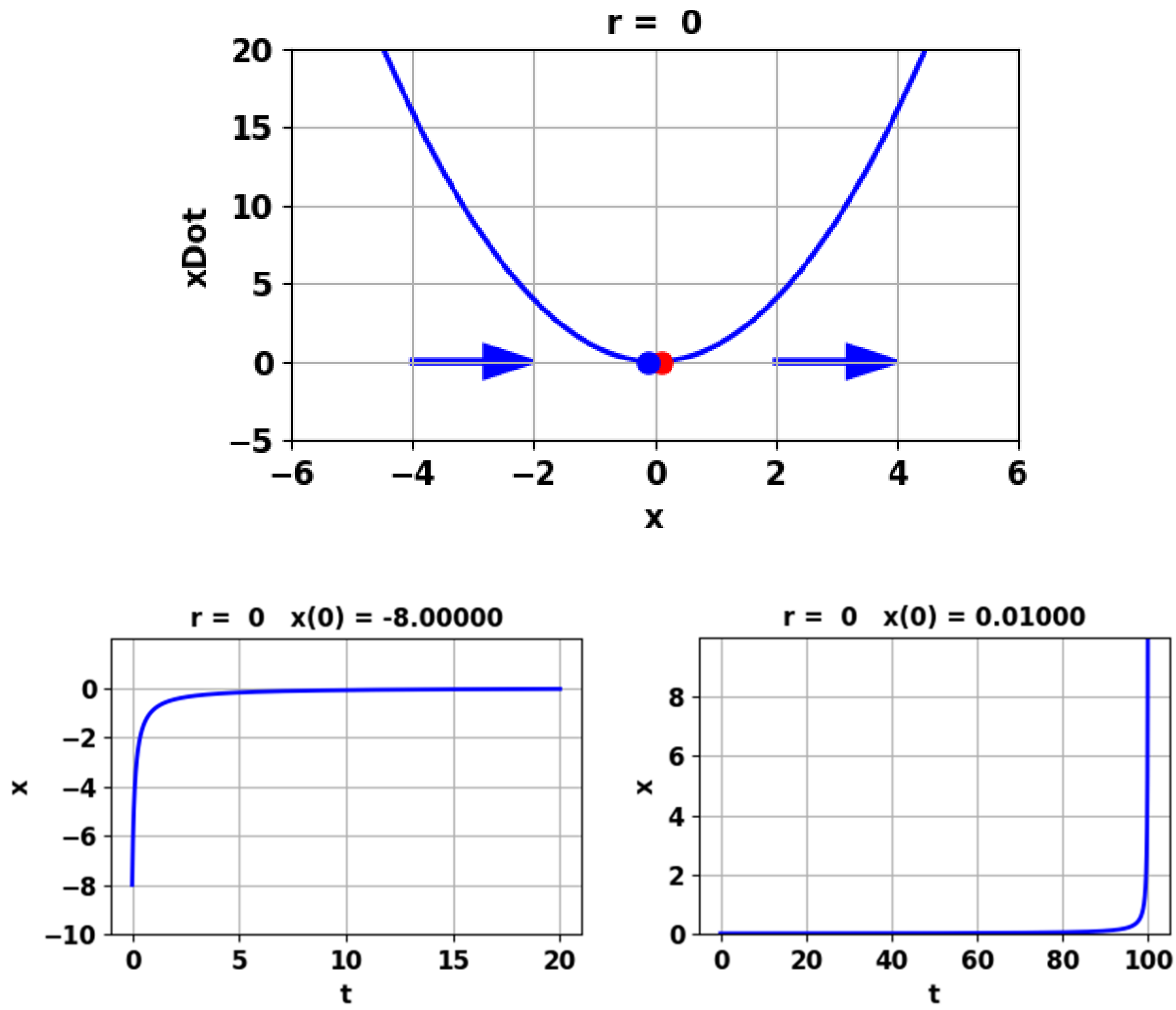
$r = 0$

$$r = 0 \quad \dot{x} = x^2 \quad x_e = 0 \quad f'(x_e = 0) = 0$$

$$x(0) = 0 \quad \dot{x}(t) = 0 \quad \Rightarrow t \to \infty \quad x \to 0$$

$$x(0) < 0 \quad \dot{x}(t) > 0 \quad \Rightarrow t \to \infty \quad x \to 0$$

$$x(0) > 0 \quad \dot{x}(t) > 0 \quad \Rightarrow t \to \infty \quad x \to +\infty$$



Fig, 1.2 Fixed point: $r = 0$, $x_e = 0$.

Blue **dot** is a stable fixed point (negative slope)

Red **dot** is an unstable fixed point (positive slope).

$r < 0$

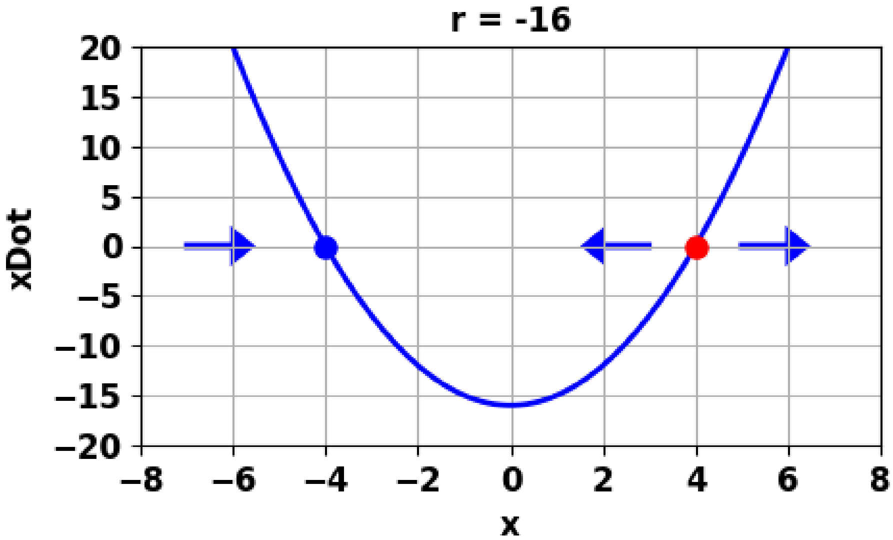
There are two fixed points

$$\dot{x} = r + x^2 \quad f(x) = r + x^2 \quad f'(x) = 2x$$

$$x_e = -\sqrt{-r} \quad f'(x_e) < 0 \quad \Rightarrow \quad \text{stable}$$

$$x_e = +\sqrt{-r} \quad f'(x_e) > 0 \quad \Rightarrow \quad \text{unstable}$$

Let r = -16 then the two fixed points are x_e = -4 (stable) and x_e = +4 (unstable).



This is a very simple system but its dynamics is highly interesting. The bifurcation in the dynamics occurred at $r = 0$ (bifurcation point), since the vector fields for $r < 0$ and $r > 0$ qualitatively different.

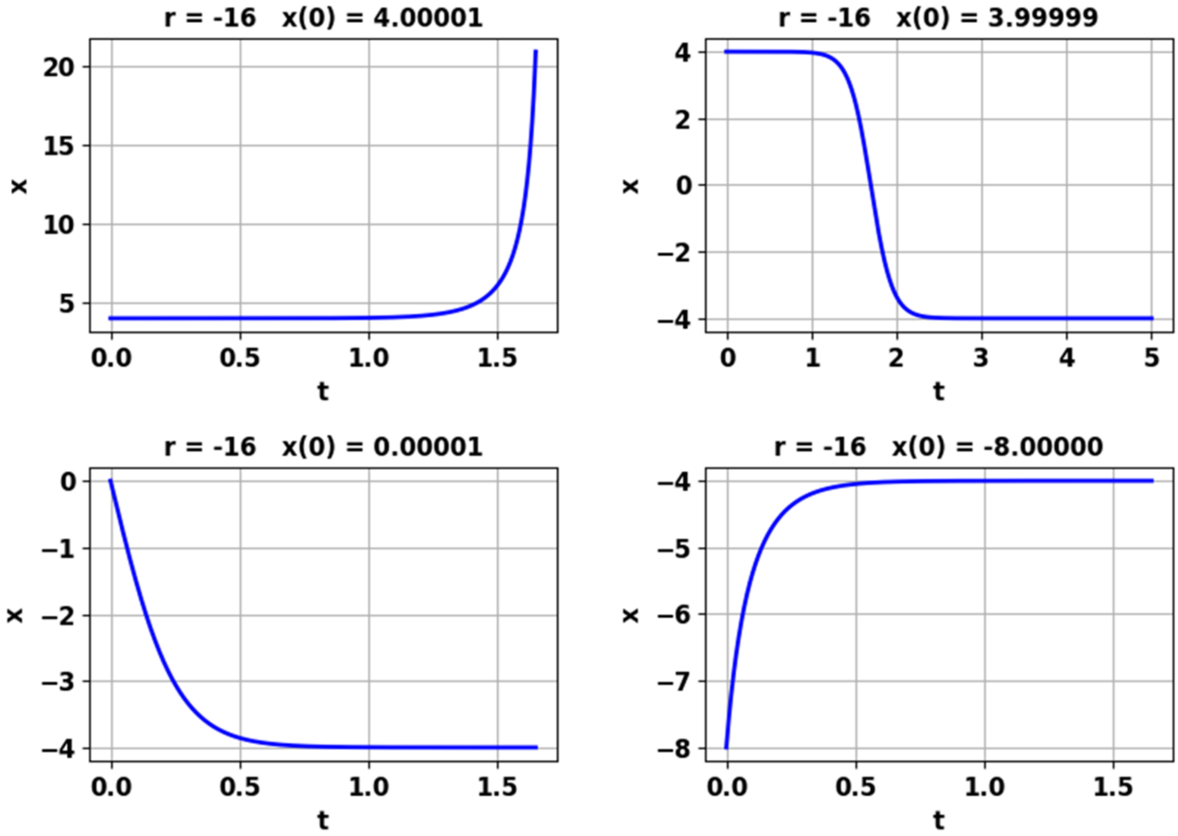


Fig. 1.3   Stable fixed point x_e = -4 (blue dot, negative slope)

Unstable fixed point x_e = +4 (red dot, positive slope)

$$x(0) > 4 \quad t \to \infty \quad \Rightarrow \; x(t) \to \infty$$

$$x(0) < 4 \quad t \to \infty \quad \Rightarrow \; x(t) \to -4$$

$$x(0) = 4 \quad t \to \infty \quad \Rightarrow \; x(t) = 4$$

Figure 1.4 shows the **bifurcation diagram** for the fixed points x_e as a function of the **bifurcation parameter *r***.

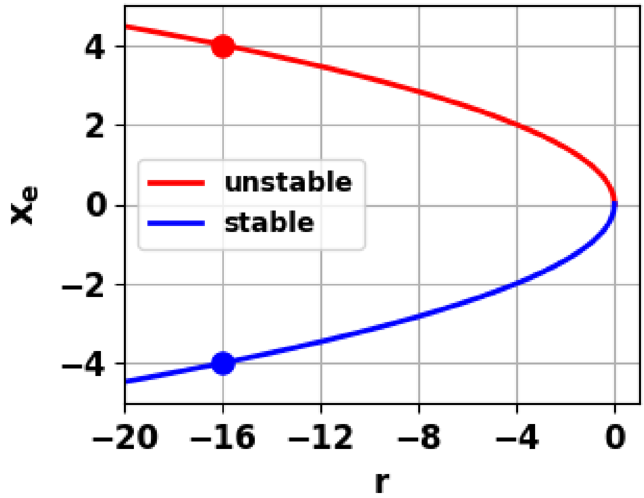


Fig. 1.4 Saddle node bifurcation diagram. The two fixed points for $r < 0$ merge as r goes to zero.

This is an example of a **subcritical saddle node bifurcation** since the fixed points exist for values of the parameter below the bifurcation point $r = 0$. $r = 0$, $x_e = 0$ is called the saddle node bifurcation point.

Start with a large negative r value. Then as r approaches zero, the two fixed points get closer together and at $r = 0$ the two fixed points coalesces and annihilate each other and as r becomes greater than zero, there are no longer any fixed points.

Sometimes a saddle node bifurcation is called a fold, or turning point, or blue-sky bifurcation.

Example 2 SUPERCRITICAL SADDLE NODE BIFURCATION

cs100super.py

$\dot{x}(t) = r - x(t)^2$ r is an adjustable constant

$$f(x) = r - x^2 \qquad f'(x) = -2x$$

$$\dot{x} = 0 \;\Rightarrow\; x_e = \pm\sqrt{r}$$

Thus, there are three possible fixed points;

$r < 0$ no fixed points

$r = 0$ one fixed point $x_e = 0$

$r > 0$ two fixed points $x_e = -\sqrt{r}$ $x_e = +\sqrt{r}$

$r < 0$ there are no fixed-points

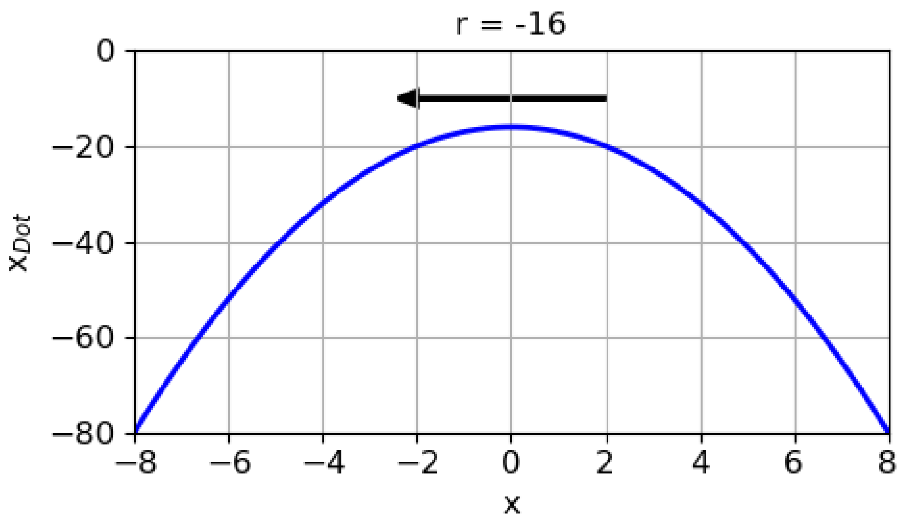


Fig. 2.1 If $r < 0$ then there are no fixed points. The flow is always to the left $t \to \infty$ $x(t) \to \infty$.

$r = 0$

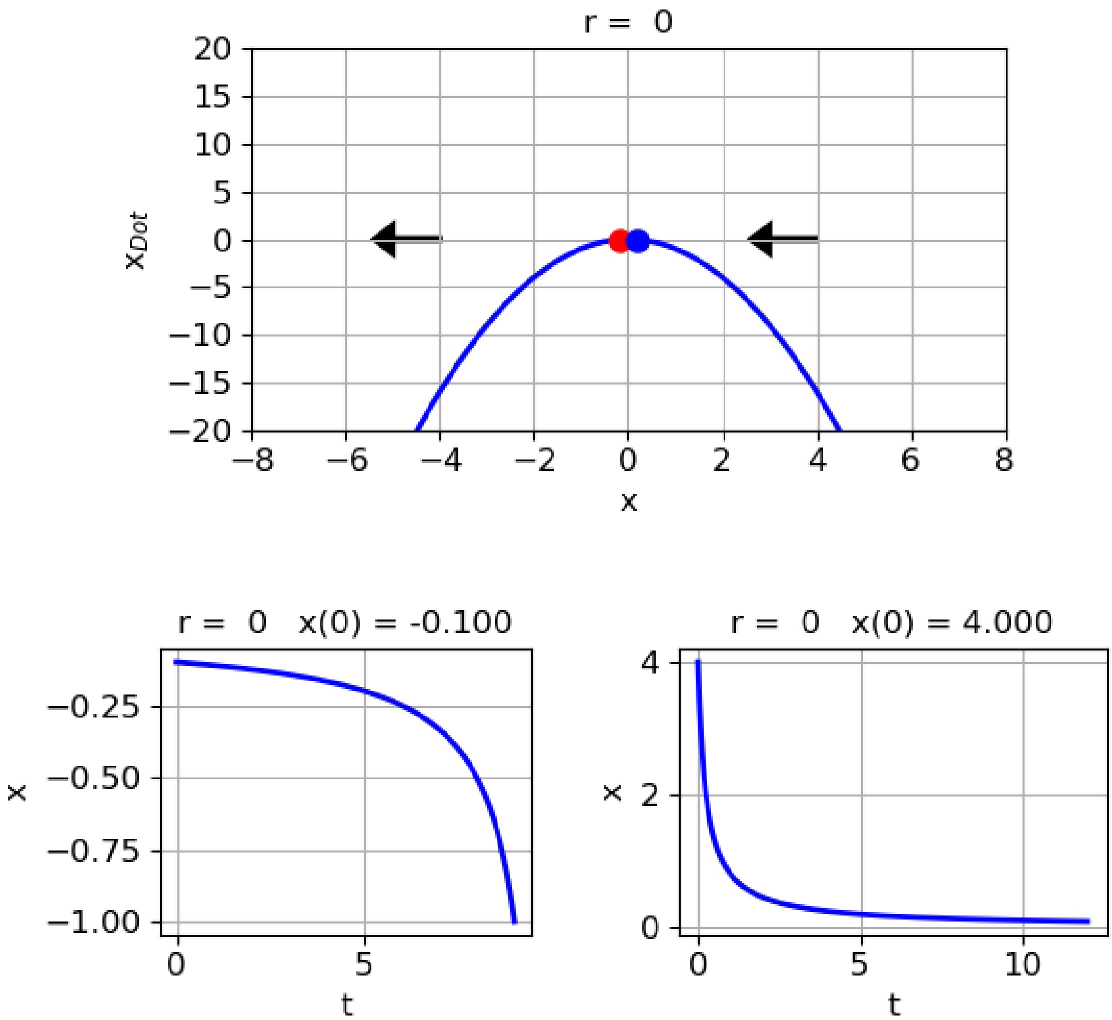
$$r = 0 \quad \dot{x} = -x^2 \quad f(x) = -x^2 \quad f'(x) = -2x$$

$$x_e = 0 \quad f'(x_e = 0) = 0$$

$$x(0) = 0 \quad \dot{x}(t) = 0 \quad \Rightarrow t \to \infty \quad x = 0$$

$$x(0) < 0 \quad \dot{x}(t) < 0 \quad \Rightarrow t \to \infty \quad x \to -\infty$$

$$x(0) > 0 \quad \dot{x}(t) < 0 \quad \Rightarrow t \to \infty \quad x \to 0$$



Fig, 2.2 Fixed point: $r = 0$, $x_e = 0$.

Blue **dot** is a stable fixed point (negative slope)

Red **dot** is an unstable fixed point (positive slope).

$r > 0$

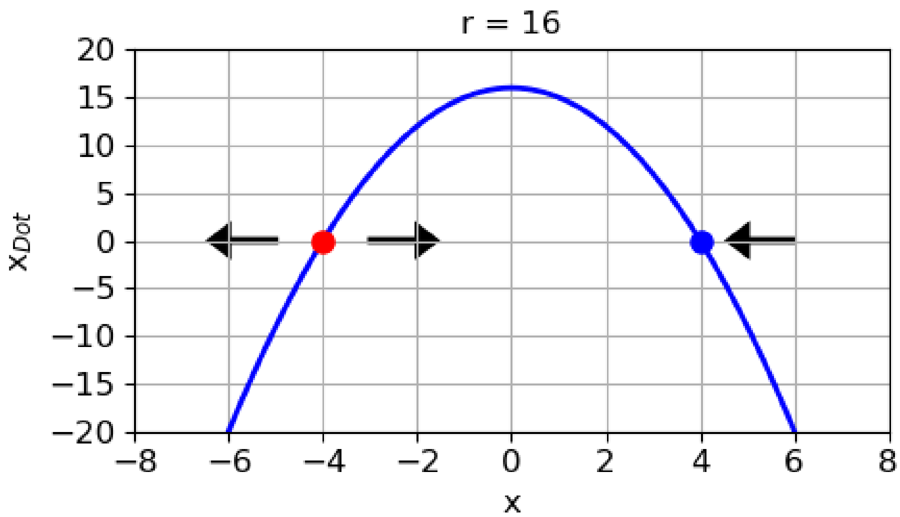
There are two fixed points

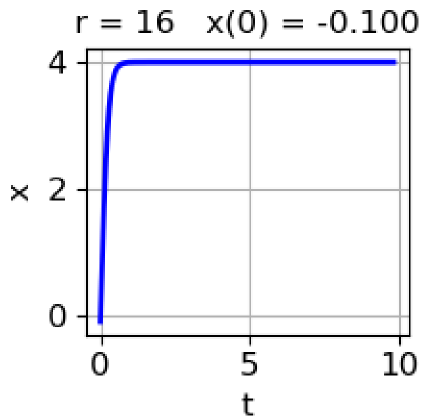
$$\dot{x} = r - x^2 \qquad f(x) = r - x^2 \qquad f'(x) = -2x$$

$$x_e = -\sqrt{r} \qquad f'(x_e) < 0 \quad \Rightarrow \quad \text{stable}$$

$$x_e = +\sqrt{r} \qquad f'(x_e) > 0 \quad \Rightarrow \quad \text{unstable}$$

Let $r = 16$ then the two fixed points are $x_e = 4$ (stable) and $x_e = -4$ (unstable).





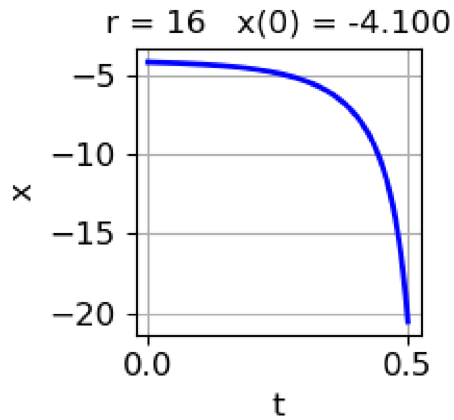


Fig. 2.3 Stable fixed point $x_e = 4$ (blue dot, negative slope)

Unstable fixed point $x_e = -4$ (red dot, positive slope)

$$x(0) > 4 \quad t \to \infty \quad \Rightarrow \; x(t) \to 4$$

$$-4 < x(0) < 4 \quad t \to \infty \quad \Rightarrow \; x(t) \to 4$$

$$x(0) < -4 \quad t \to \infty \quad \Rightarrow \; x(t) \to -\infty$$

Figure 2.4 shows the **bifurcation diagram** for the fixed points x_e as a function of the **bifurcation parameter *r***.

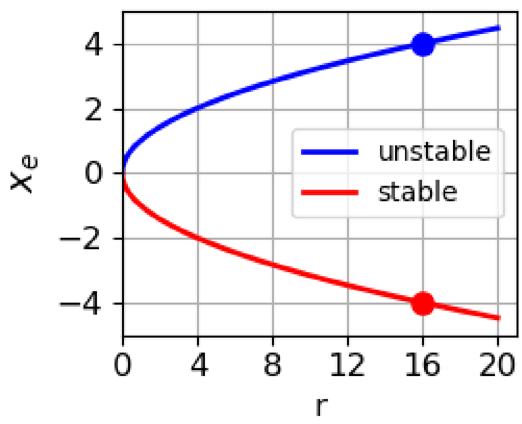


Fig. 2.4 Saddle node bifurcation diagram. The two fixed points for $r > 0$ merge as r goes to zero.

This is an example of a **supercritical saddle node bifurcation** since the fixed points exist for values of the parameter above the bifurcation point $r = 0$.

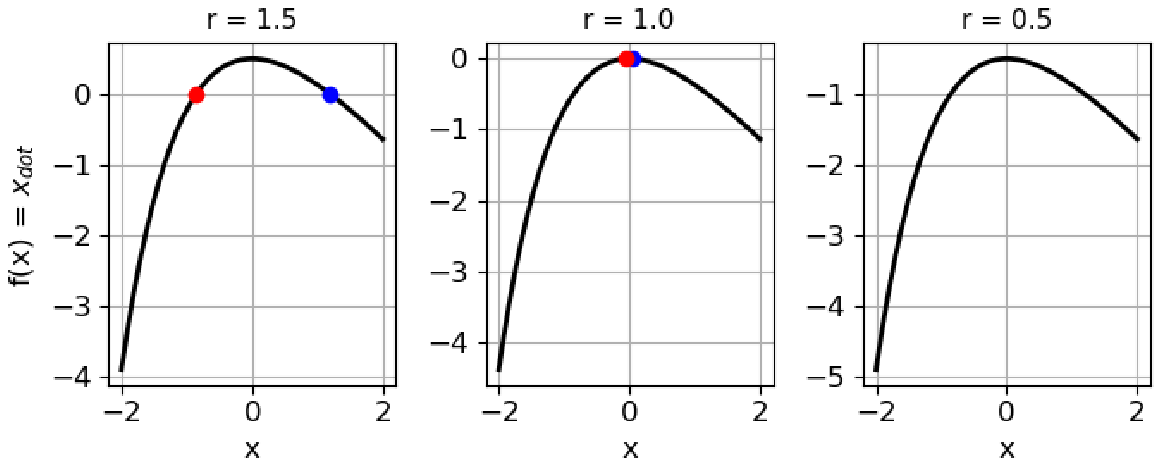
The normal form for a subcritical bifurcation is $\dot{x}(t) = r + x(t)^2$

Other ODEs in a local neighbourhood can also be considered as a saddle node bifurcation. For example,

$$\dot{x} = r - x - e^{-x}$$

where r is the bifurcation parameter and the bifurcation point is $r = 1$.

The function x vs $\dot{x}$ is plotted using the Python code **cs101B.py**.



When the slope at a fixed point is positive, the flow is to the right $\rightarrow$, and when negative, the flow is to the left $\leftarrow$. So, near a **stable fixed point**, the flow is always towards it, and always away from an **unstable fixed point**.

Decreasing the r value:

- $r > 1$ two distinct fixed points (stable and unstable).
- $r = 1$ fixed points merge to give a single saddle node fixed point
- $r < 1$ the fixed points are annihilated.

Increasing the r value:

$r < 1$	no fixed points exist.
$r = 1$	a fixed point is created as a single saddle node fixed point
$r > 1$	two distinct fixed points created (stable and unstable).

Reference

https://math.libretexts.org/Bookshelves/Scientific_Computing_Simulations_and_Modeling/Scientific_Computing_(Chasnov)/II%3A_Dynamical_Systems_and_Chaos/12%3A_Concepts_and_Tools