DOING PHYSICS WITH PYTHON

DYNAMICAL SYSTEMS [1D]

Transcritical Bifurcations

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Google drive

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cs101.py   **cs101A.py**

Jason Bramburger

Transcritical Bifurcations - Dynamical Systems | Lecture 7

https://www.youtube.com/watch?v=m5vmc_HxMFs&t=3s

INTRODUCTION

This lecture focuses on transcritical bifurcations. These bifurcations are characterized by two fixed points colliding and exchanging stability. Unlike saddle-node bifurcations, no fixed points are created or destroyed over the course of the bifurcation.

The [1D] nonlinear system's ODE can be expressed as

$$\dot{x}(t) = f\left(x(t), r\right)$$

and the fixed points of the system are

$$f\left(x_e(t), r\right) = 0$$

where r is the bifurcation parameter. So, the fixed points x_e and their stability depends upon the bifurcation parameter.

A transcritical bifurcation occurs when there is an exchange of stabilities between two fixed points. The **normal form** for a **transcritical bifurcation** is given by

$$\dot{x}(t) = r\,x(t) - x(t)^2 \qquad r \text{ is an adjustable constant}$$

$$f(x) = r\,x - x^2 \qquad f'(x) = r - 2\,x$$

***r* = 0**

$$\dot{x} = -x^2 \Rightarrow \; x_e = 0 \quad f(x) = -x^2 \quad f'\left(x_e\right) = -2x_e = 0 \Rightarrow$$

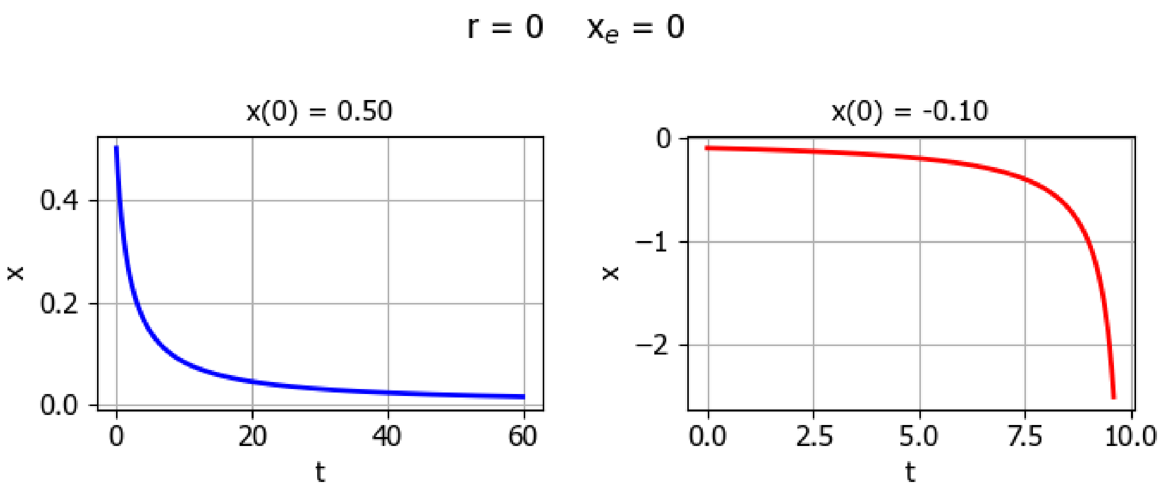
System has only **one** equilibrium point at $\boldsymbol{x_e = 0}$ and its stability is inconclusive from $f'\left(x_e\right) = 0$.

The flow is always in a negative sense (towards the left)

$$x < 0 \Rightarrow \dot{x} < 0 \Rightarrow t \to \infty \;\; x \to -\infty$$

$$x > 0 \Rightarrow \dot{x} < 0 \Rightarrow t \to \infty \;\; x \to x_e = 0$$

The equilibrium points $x_e = 0$ is an unstable saddle point.



$$x < 0 \Rightarrow \dot{x} < 0 \Rightarrow t \to \infty \;\; x \to -\infty$$

$$x > 0 \Rightarrow \dot{x} < 0 \Rightarrow t \to \infty \;\; x \to x_e = 0$$

For $r \neq 0$, there are two distinct equilibrium, $x_e = 0$ and $x_e = r$.

$r < 0$

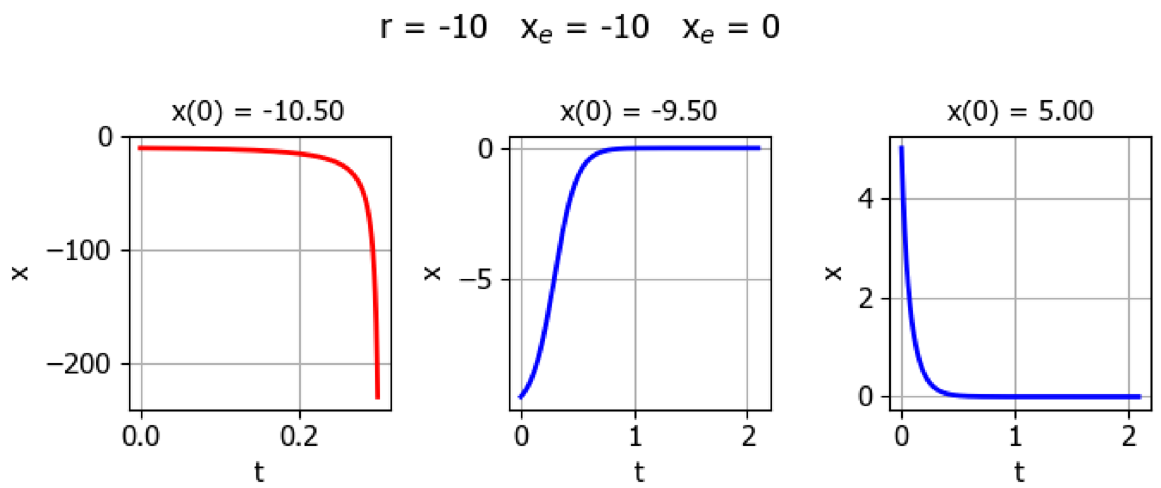
$$\dot{x}\big|_{x_e} = r\,x_e - x_e^{\,2} = 0 \qquad f'\left(x_e\right) = r - 2x_e$$

$$x_e = 0 \quad f'\left(x_e\right) = f'(0) = r < 0 \;\; \Rightarrow$$

equilibrium point at the Origin $x_e = 0$ is **stable** (**sink**).

$$x_e = r \quad f'\left(x_e\right) = f'(r) = -r > 0 \;\; \Rightarrow$$

equilibrium point $x_e = r$ is **unstable** (**source**).



equilibrium point $x_e = 0$ is **stable** (**sink**)

equilibrium point $x_e = r$ is **unstable** (**source**).

***r* > 0**

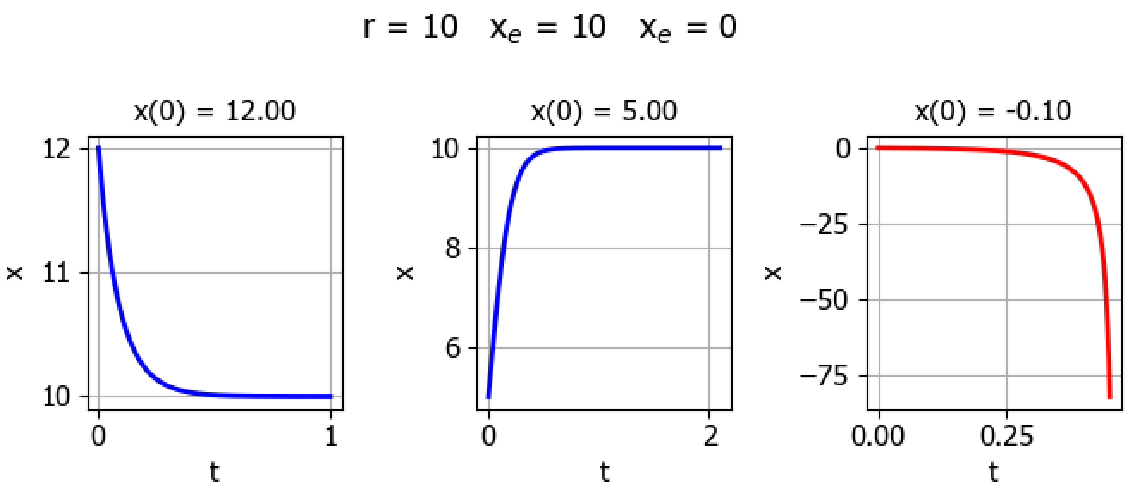
$$\dot{x}\big|_{x_e} = r\,x_e - x_e^{\,2} = 0 \qquad f'(x_e) = r - 2x_e$$

$$x_e = 0 \quad f'(x_e) = f'(0) = r > 0 \;\Rightarrow$$

equilibrium point at the Origin $x_e = 0$ is **unstable** (**source**).

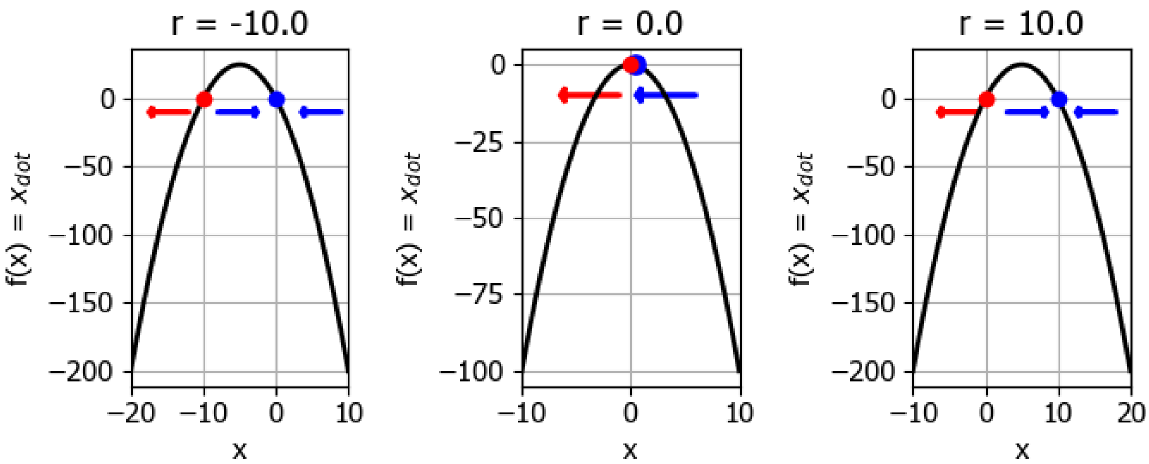
$$x_e = r \quad f'(x_e) = f'(r) = -r < 0 \;\Rightarrow$$

equilibrium point $x_e = -10$ is **stable** (**sink**).

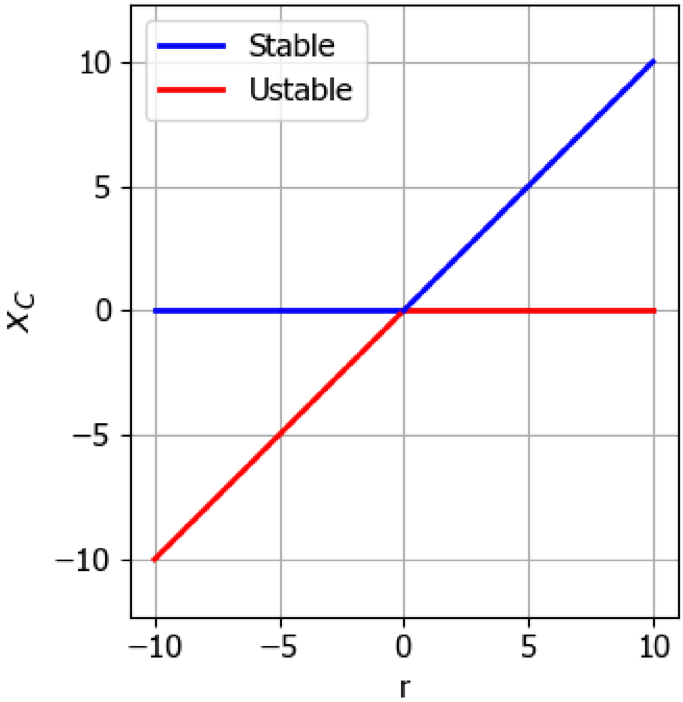


equilibrium point x_e =10 is **stable** (**sink**)

equilibrium point $x_e = 0$ is **unstable** (**source**).



As r increases from -10 to 0 to 10, the two fixed points move towards each other, at $r = 0$, they merge and then for $r > 0$ they separate again with exchanged stabilities. The transcritical bifurcation point is $r = 0$.



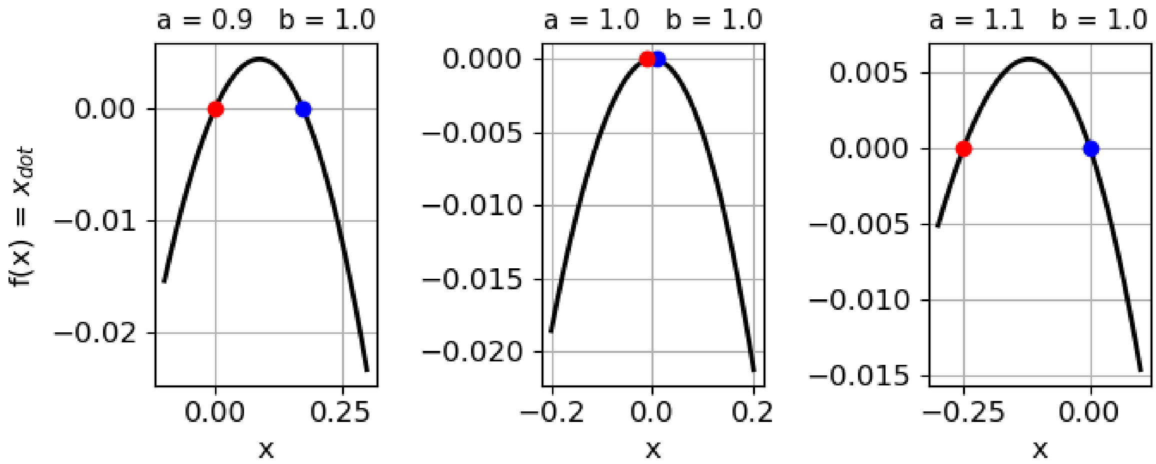
This type of bifurcation diagram is known as a transcritical bifurcation. In this bifurcation, an exchange of stabilities has taken place between the two fixed points of the system.

Other ODEs can be reduced to the normal form of a transcritical bifurcation in a local region. As an example, consider the ODE

$$\dot{x} = (1 - x)\,x - a\left(1 - e^{-b\,x}\right)$$

Then in the locality of $a\,b = 1$, this ODE approximates the normal form of the trancritical bifurcation $\dot{x} = r\,x - x^2$.

A plot of $\dot{x}$ vs x tells us all we need to know about the trajectories of the flow when $a\,b \sim 1$. Let $b = 1$ and a be the bifurcation parameter with values 0.9 ($ab = 0.9$), 1.0 ($ab = 1.0$), and a =1.1 ($ab = 1.1$) as shown in the following plots.



When the slope at a fixed point is positive, the flow is to the right →, and when negative, the flow is to the left ←. So, near a **stable fixed point**, the flow is always towards it, and always away from an **unstable fixed point**. **cs101A.py**

Reference

https://math.libretexts.org/Bookshelves/Scientific_Computing_Simulations_and_Modeling/Scientific_Computing_(Chasnov)/II%3A_Dynamical_Systems_and_Chaos/12%3A_Concepts_and_Tools