DOING PHYSICS WITH PYTHON

DYNAMICAL SYSTEMS [1D]

Pitchfork Bifurcations

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DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive

GitHub

cs103.py **cs104.py** **cs105.py**

Jason Bramburger

Pitchfork Bifurcations - Dynamical Systems | Lecture 8

https://www.youtube.com/watch?v=rkXHEsn-DQ4

INTRODUCTION

This lecture focuses on pitchfork bifurcations. A pitchfork bifurcation is a particular type of local bifurcation (possible in dynamical systems) that has symmetry. In such cases, equilibrium points appear and disappear in symmetrical pairs. The bifurcation diagram looks like a pitchfork, hence the name pitchfork bifurcation. There are two types of pitchfork bifurcations, namely supercritical and subcritical. A pitchfork bifurcation is called supercritical if a stable solution branch bifurcates into two new stable branches as the parameter r is increased. It is called subcritical if two unstable and one stable equilibria collapse to produce one stable one.

Example 1 Supercritical pitchfork bifurcation cs103.py

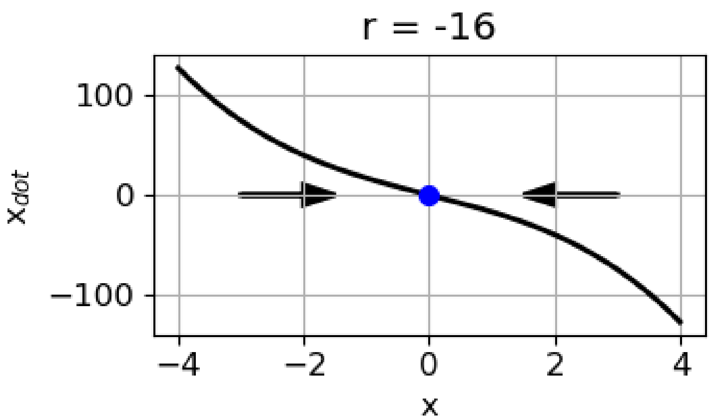
$\dot{x}(t) = r\,x(t) - x(t)^3$ r is an adjustable constant

$f(x,r) = r\,x - x^3 \quad f\,'(x,r) = r - 3x^2$

The system is invariant under the transformation

$$x \to -x \quad r(-x) - (-x)^3 = -\left(rx - x^3\right) = -\dot{x}$$

$r < 0$ one fixed point $x_e = 0$

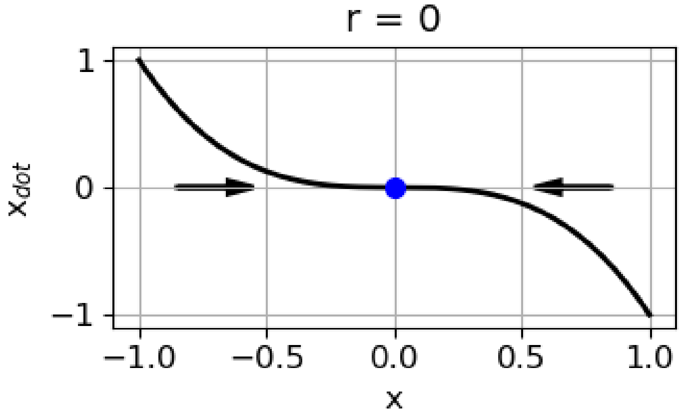


$r < 0 \quad \dot{x} = 0 \Rightarrow x_e = 0 \quad f\,'(0) < 0$ stable

$x(0) < 0 \quad t \to \infty \quad x(t) \to 0$

$x(0) > 0 \quad t \to \infty \quad x(t) \to 0$

$r = 0$ one stable fixed point **$x_e = 0$**



$$r = 0 \quad \dot{x} = 0 \Rightarrow x_e = 0 \quad f'(0) = 0 \quad \text{stable}$$

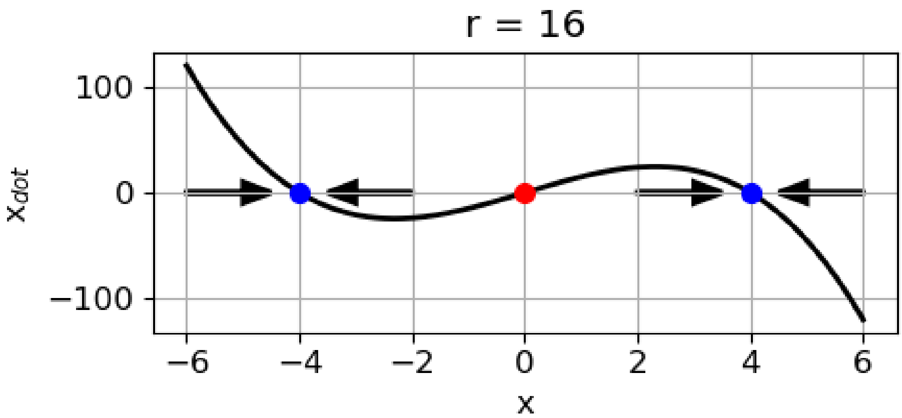
$$x(0) < 0 \quad t \to \infty \quad x(t) \to 0$$

$$x(0) > 0 \quad t \to \infty \quad x(t) \to 0$$

$r > 0$ three fixed points

$$\dot{x} = 0 \quad x_e = 0 \quad f'(0) = r > 0 \quad \text{unstable}$$

$$\dot{x} = 0 \quad x_e = \pm\sqrt{r} \quad f'(\pm\sqrt{r}) = -2r < 0 \quad \text{stable}$$

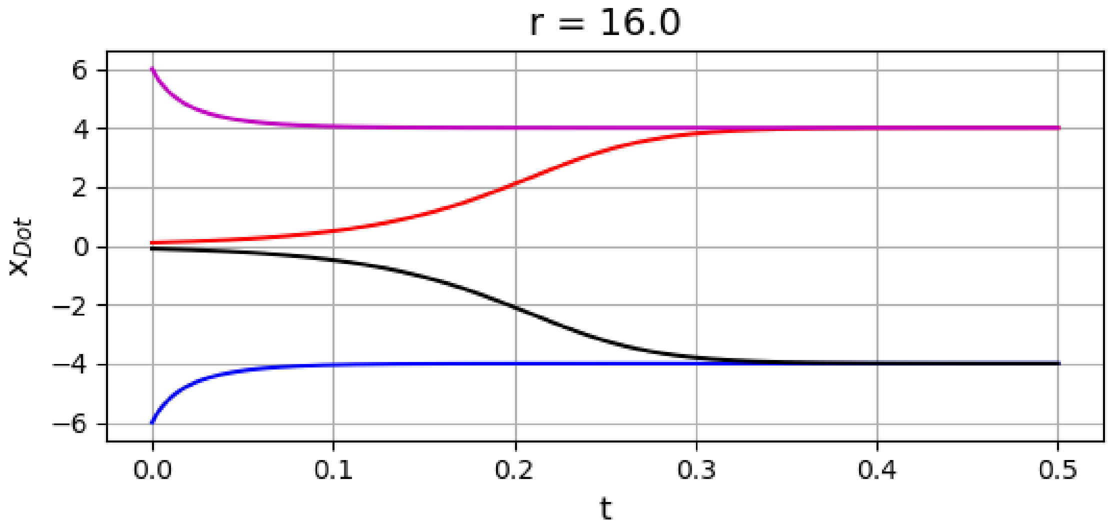


$$x(0) < -4 \quad t \to \infty \quad x(t) \to -4$$

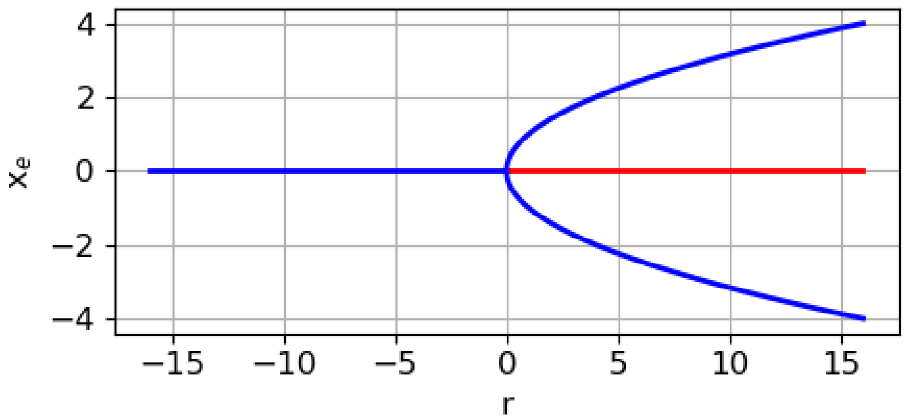
$$-4 < x(0) < 0 \quad t \to \infty \quad x(t) \to -4$$

$$0 < x(0) < 4 \quad t \to \infty \quad x(t) \to +4$$

$$x(0) > 4 \quad t \to \infty \quad x(t) \to +4$$



Time evolution of the flow for different initial conditions when $r = 16.0 > 0$.



Supercritical pitchfork bifurcation

$r = 0$ one fixed point: $x_e = 0$ stable

$r < 0$ one fixed point: $x_e = 0$ stable

$r > 0$ three fixed points: $x_e = 0$ unstable

$x_e = \pm\sqrt{r}$ stable

The pitchfork bifurcations occur when one fixed point becomes three at the bifurcation point. Pitchfork bifurcations are usually associated with the physical phenomena called symmetry breaking. For the **supercritical pitchfork bifurcation**, the stability of the original fixed point changes from stable to unstable and a new pair of stable fixed points are created above and below the bifurcation point.

From the pitchfork-shape bifurcation diagram, the name 'pitchfork' becomes clear. But it is basically a pitchfork trifurcation of the system. The bifurcation for this vector field is called a supercritical pitchfork bifurcation, in which a stable equilibrium bifurcates into two stable equilibria.

Example 2 Subcritical pitchfork bifurcation cs104.py

The transformation $x \to -x$, gives the subcritical pitchfork bifurcation $\left(\ddot{x} = rx + x^3\right)$ as shown in the following example.

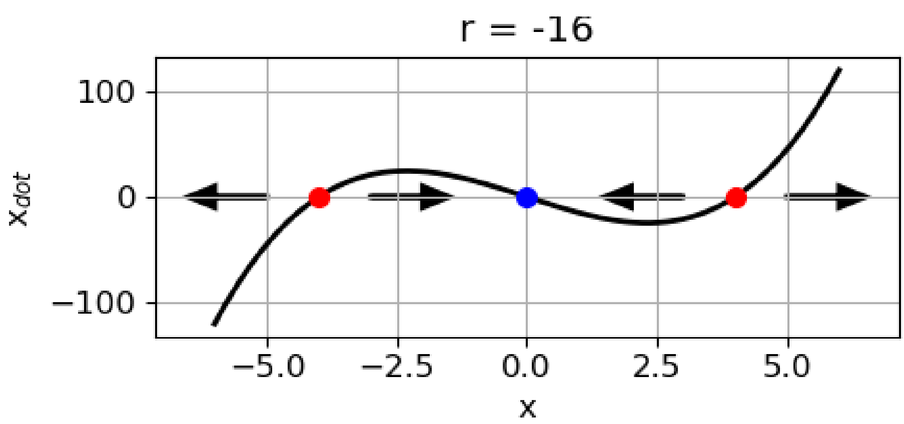
$\dot{x}(t) = r\,x(t) + x(t)^3$ r is an adjustable constant

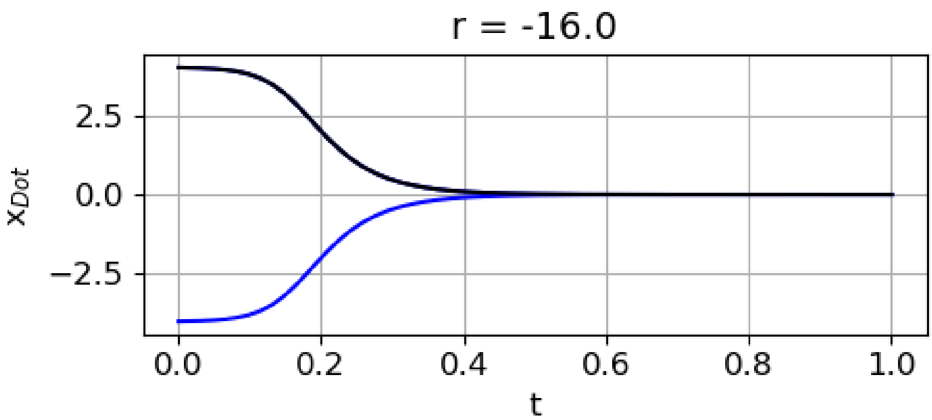
$$f(x) = r\,x + x^3 \quad f\,'(x) = r + 3\,x^2$$

$r < 0$ three fixed points

$$\dot{x} = 0 \quad x_e = 0 \quad f\,'(0) = r < 0 \quad \textbf{stable}$$

$$r < 0 \quad \dot{x} = 0 \quad x_e = \pm\sqrt{-r} \quad f\,'(\pm\sqrt{-r}) = -2\,r > 0 \quad \textbf{unstable}$$





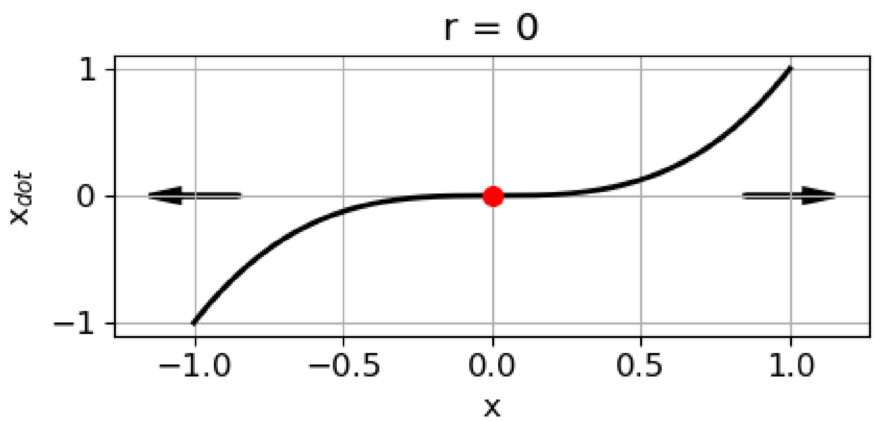
$$x(0) = 3.99 \text{ and } x(0) = -3.99$$

$\boldsymbol{r = 0}$ one fixed point

$$\dot{x} = 0 \Rightarrow x_e = 0 \quad f'(0) = 0 \quad \textbf{unstable}$$

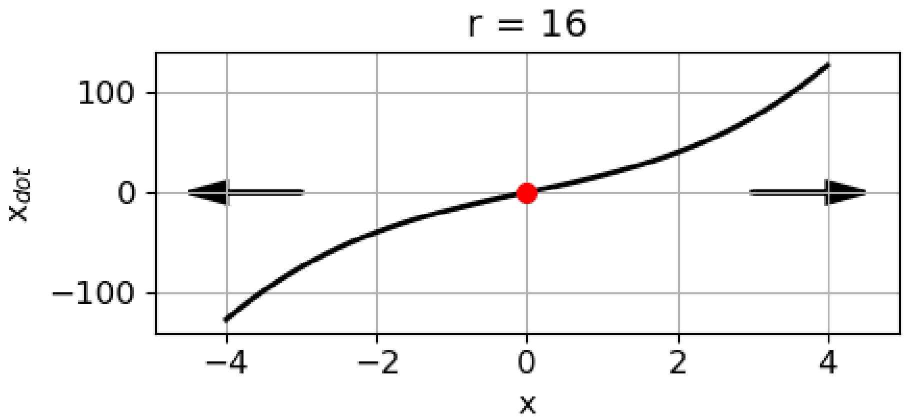
$$x(0) < 0 \quad \dot{x}(0) < 0 \quad t \to \infty \quad x(t) \to -\infty$$

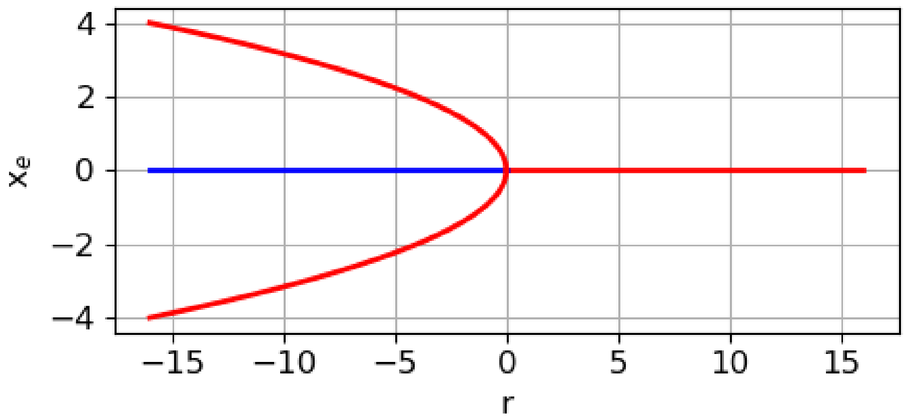
$$x(0) > 0 \quad \dot{x}(0) > 0 \quad t \to \infty \quad x(t) \to +\infty$$



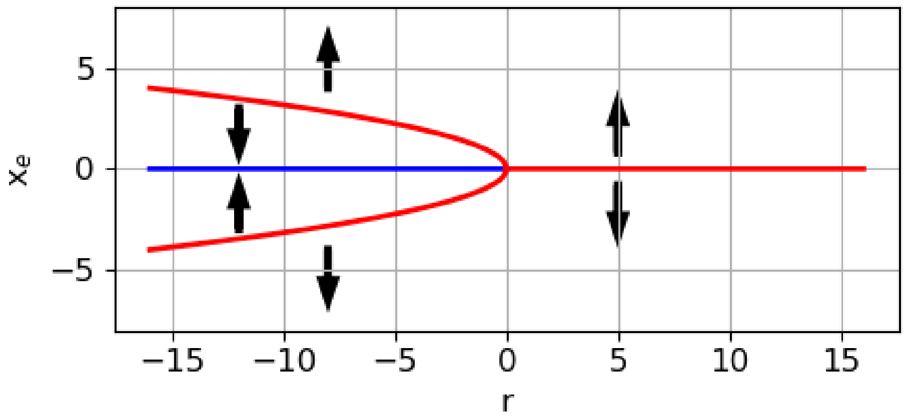
$r > 0$ one fixed point

$\dot{x} = 0 \Rightarrow x_e = 0 \quad f'(0) = r > 0$ **unstable**





Bifurcation diagram: bifurcation parameter is r and the bifurcation point is $r = 0$.



Bifurcation diagram and flow along the line: the flow is aways directed towards a stable fixed point and away from an unstable fixed point.

$x_e = 0$ is unstable for $r \geq 0$ $\quad \leftarrow x_e \rightarrow$

$x_e = 0$ is stable for $r < 0$ $\quad \rightarrow x_e \leftarrow$

$x_e \neq 0$ is unstable for r < 0 $\quad \leftarrow x_e \rightarrow$

Example 3 $\dot{x}(t) = r\,x(t) + x(t)^3 - x(t)^5$ **cs105.py**

$\dot{x} = r\,x + x^3 - x^5$ r is an adjustable constant

$f(x) = r\,x + x^3 - x^5 \quad f'(x) = r + 3x^2 - 5x^4$

$$\dot{x} = 0 \quad \Rightarrow \quad x_e\left(r + x_e^{\,2} - x_e^{\,4}\right) = 0$$

$$x_e = 0 \quad -x_e^{\,4} + x_e^{\,2} + r = 0$$

$$+z^2 - z - r = 0 \qquad z = x_e^{\,2}$$

$$z = \tfrac{1}{2}\left(1 \pm \sqrt{1+4r}\right)$$

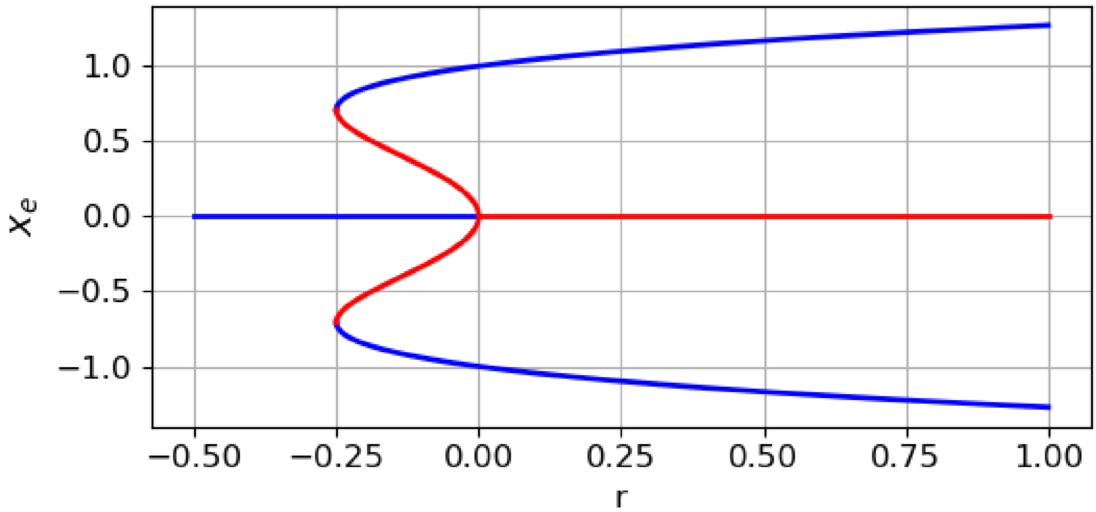
$$x_e = \pm\sqrt{\tfrac{1}{2}\left(1 \pm \sqrt{1+4r}\right)}$$

$$f'(x_e) = r + 3x_e^{\,2} - 5x_e^{\,4}$$

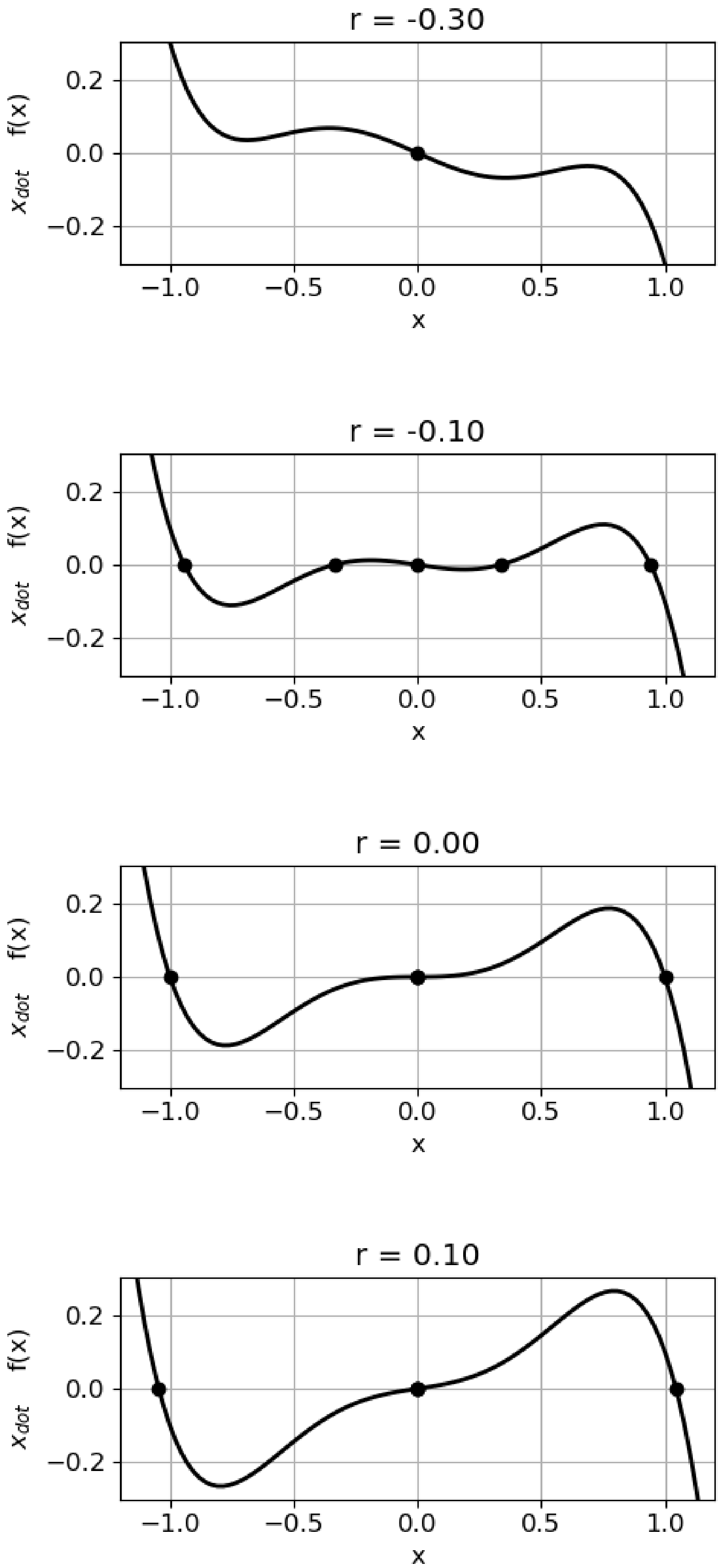
The bifurcation diagram shown below has in addition to a subcritical pitchfork bifurcation at the Origin, and two symmetric saddle node bifurcations that occur when $r = -1/4$. We can imagine what happens to the solution x(t) as r increases from negative values, assuming there is some noise in the system so that $x(t)$ fluctuates around a stable fixed point. For $r < -1/4$, the solution $x(t)$ fluctuates around the stable fixed point $x_e = 0$. As r increases into the range $-1/4 < r < 0$, the solution will remain close to the stable fixed point $x_e = 0$. However, a catastrophic event occurs as soon as $r > 0$. The fixed point $x_e = 0$ is lost and the solution will jump up or down to one of the fixed points. A similar catastrophe can happen as r decreases from positive values. In this case, the jump occurs as soon as $r < -1/4$

Since the behaviour of $x(t)$ is different depending on whether we increase or decrease r, we say that the system exhibits **hysteresis**.

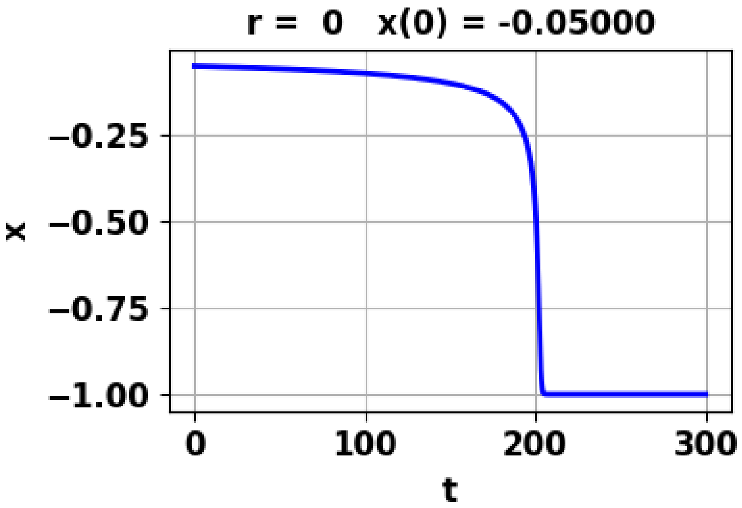
The existence of a subcritical pitchfork bifurcation can be very dangerous in engineering applications since a small change in the physical parameters of a problem can result in a large change in the equilibrium state. Physically, this can result in the collapse of a structure.

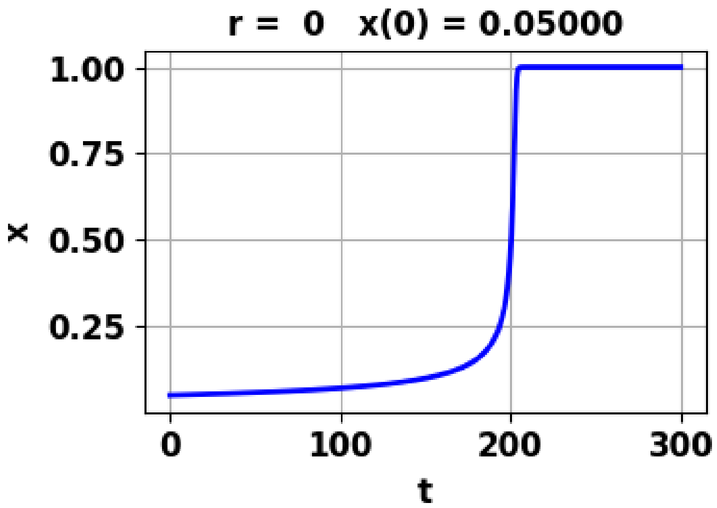


Subcritical pitchfork bifurcation at the Origin, and two symmetric saddle node bifurcations that occur when $r = -1/4$. In a local neighbourhood, the flow is always towards a stable fixed point and away from an unstable fixed point.

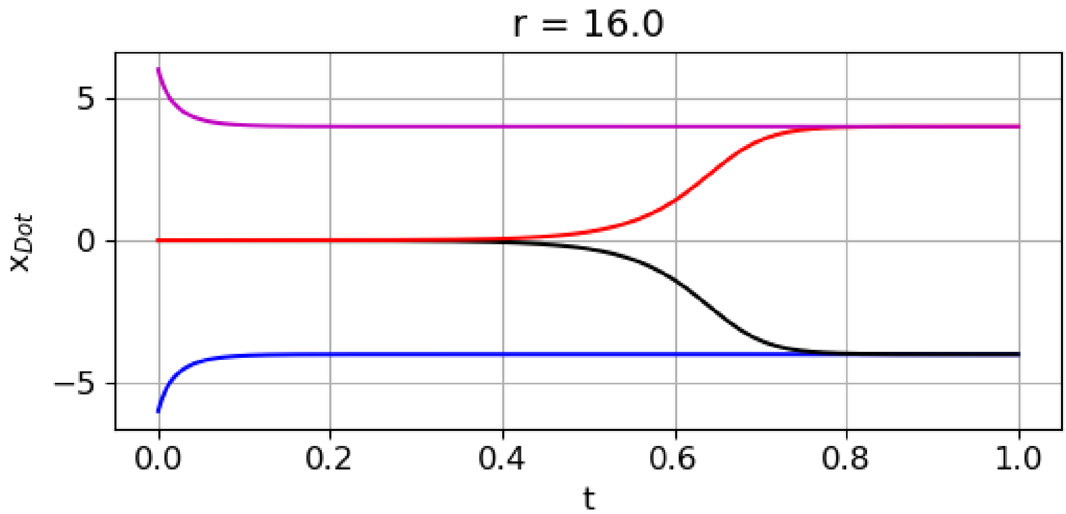


At a **fixed point**: negative slope → **stable**, positive slope → **unstable**





Slight differences in the initial conditions can lead to dramatic differences in the time evolution of the flow and steady state value for x.



$x(0) = +\,6$ $x(0) = -\,6$ $x(0) = +\,0.00001$ $x(0) = -\,0.00001$

You see that our system is extremely sensitive to the initial conditions. Although the system is deterministic, the system is not completely predictable for initial conditions near $x_e = 0$. In this instance, you cannot make useful predictions since unmeasurable differences in the initial conditions lead to dramatically different outcomes.

$\Rightarrow$ butterfly effect

The idea that a mathematical equation gave you the power to predict how a system will behave is ***dead – end of the Newtonian dream***.

Reference

https://math.libretexts.org/Bookshelves/Scientific_Computing_Simulations_and_Modeling/Scientific_Computing_(Chasnov)/II%3A_Dynamical_Systems_and_Chaos/12%3A_Concepts_and_Tools