DOING PHYSICS WITH PYTHON

DYNAMICAL SYSTEMS [1D]

Imperfect Pitchfork Bifurcations

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Google drive

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ds27L9.py

Jason Bramburger

Imperfect Bifurcations - Dynamical Systems | Lecture 9

https://www.youtube.com/watch?v=C5ZfoEUPqPc

INTRODUCTION

Symmetry plays a critical role in pitchfork bifurcations. But what about when that symmetry is broken? The result is a kind of imperfect bifurcation. In this article, we study a specific example of an imperfect bifurcation, sometimes called a cusp bifurcation, and show what happens when the symmetry of a pitchfork is broken. The system under consideration has two parameters, one bifurcation parameter and one imperfection parameter, resulting in a significant jump in complexity from the previously studied bifurcations.

Example $\dot{x} = h + r\,x - x^3$

(1) $\dot{x} = h + r\,x - x^3$

where r is the bifurcation parameters and h is the imperfection parameter.

When $h = 0$ we get the supercritical pitchfork bifurcation and when $h \neq 0$ there is a loss in symmetry.

For [1D] system the most important point of interest are the location and stability of the fixed points. The fixed points x_e can be found by solving the cubic equation using the Python function **roots**

(2) $-x_e^{\,3} + r\,x_e + h = 0 \qquad coefficients = [-1, 0, r, h]$

```
coeff = [-1,0,r,h]
Z = np.roots(coeff)
```

Only those values of Z where imaginary(Z) = 0 corresponds to a root of equation 2.

The stability of a fixed point can be determined from:

$$f = h + r\,x - x^3$$

$$df/dx = r - 3x^2$$

$$df/dx\big|_{x_e} = r - 3{x_e}^2$$

$$df/dx\big|_{x_e} < 0 \Rightarrow \text{ stable}$$

$$df/dx\big|_{x_e} > 0 \Rightarrow \text{ unstable}$$

For the plot x vs $\dot{x}$, when the slope $df/dx\big|_{x_e}$ at the fixed point x_e is **negative** it is **stable** and when **positive** it is **unstable** (figure 2).

We can find the fixed points x_e by considering the relationships

(3A) $y_1 = y_2$

(3B) $y_1 = -h$

(3C) $y_2 = r\,x_e - {x_e}^3$

The intersection of the cubic polynomial y_2 and the horizontal straight lines y_1 (figure 1) give the fixed points of the system.

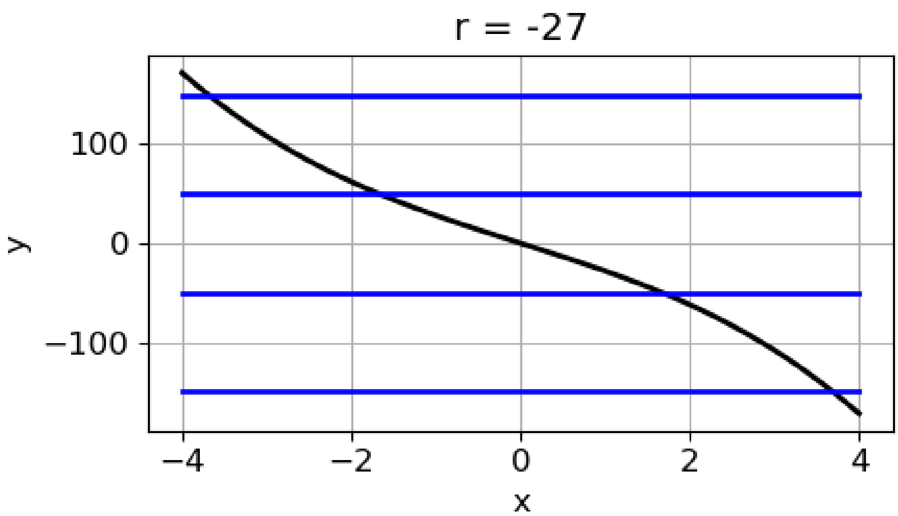


Fig. 1A. If $r \le 0$ then there is only **one** fixed point which is the intersection of the two functions $\boldsymbol{y_1}$ and $\boldsymbol{y_2}$ for any h value.

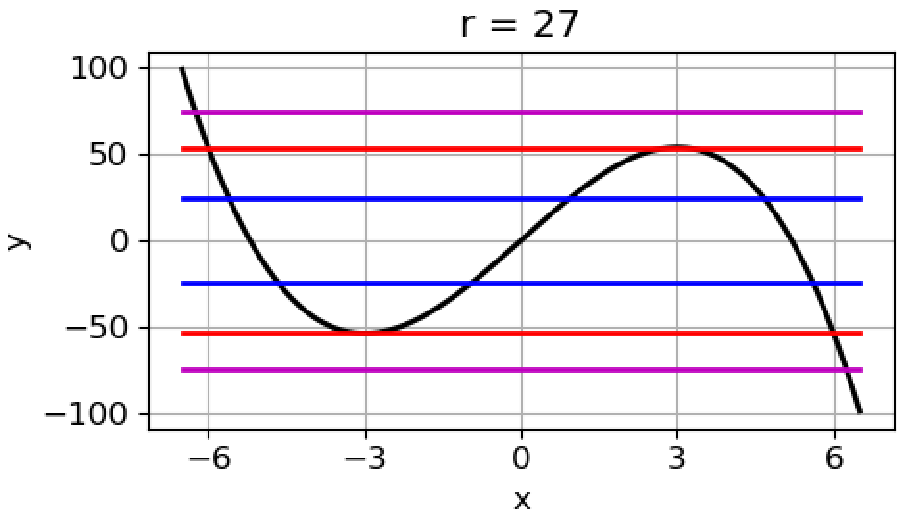


Fig. 1B. For $r > 0$, then there are three possibilities: 1 fixed point, 2 fixed points or three fixed points. The cubic polynomial (equation 3C) is antisymmetric about $x = 0$ since $x \to -x \Rightarrow y_2 \to -y_2$.

The straight lines are $y_1 = -h$.

rC = 27.00   hC = -54.00000   xC = 3.00

rC = 27.00   hC = 54.00000   xC = -3.00

Critical values: **two fixed points** $r = 27 \quad |x_C| = 3 \quad |h_C| = 54$

To determine the number of fixed points when $r > 0$, we need to find the critical values of h_C and r_C at the turning points of the cubic function y_2 where $y_1 = h_C$ is tangential to y_2. For the critical values there are two fixed points as shown in Fig. 1B (***y_1***).

The turning points (maximum and minimum) for the cubic function y_2 given by equation 3C are

(4A) $dy_2 / dx = r - 3x^2 = 0 \Rightarrow x_C = \pm\sqrt{r_C / 3} \quad r_C > 0$

and the height of the turning point h_C is

$$-h_C = r_C\, x_C - x_C{}^3$$

(4B) $$|h_C| = 2\left(\frac{r_C}{3}\right)^{3/2}$$

$|h| > |h_C|$ one fixed point

$\Downarrow$ saddle node bifurcation (splitting of fixed point)

$|h| = |h_C|$ two fixed points

$|h| < |h_C|$ three fixed point

We can find the fixed points and their stability from plots of x vs $\dot{x}$

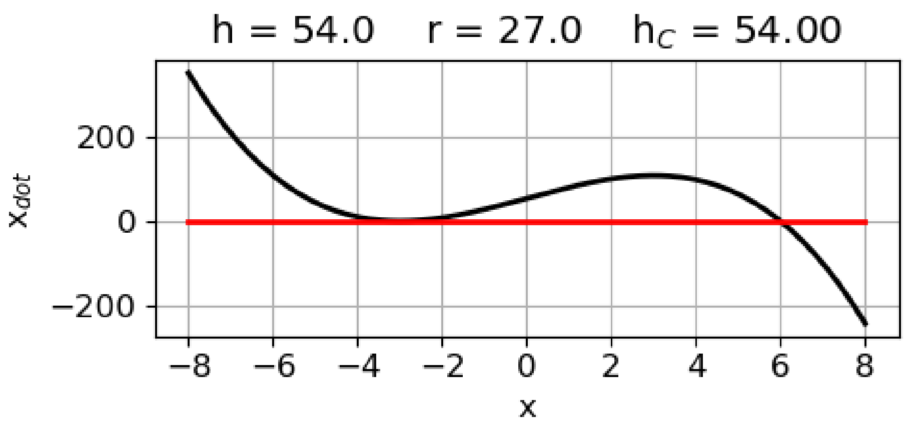


Fig. 2A. $h = h_C$

Critical values: **two fixed points** $r = 27$ $|x_C| = 3$ $|h_C| = 54$

The fixed point x_e = -3 is marginally stable and x_e = +6 is stable.

h = 53.999999 r = 27.00000
Number of fixed points = 3
xe = 6.000 stable xe = -3.000 stable
xe = -3.000 unstable

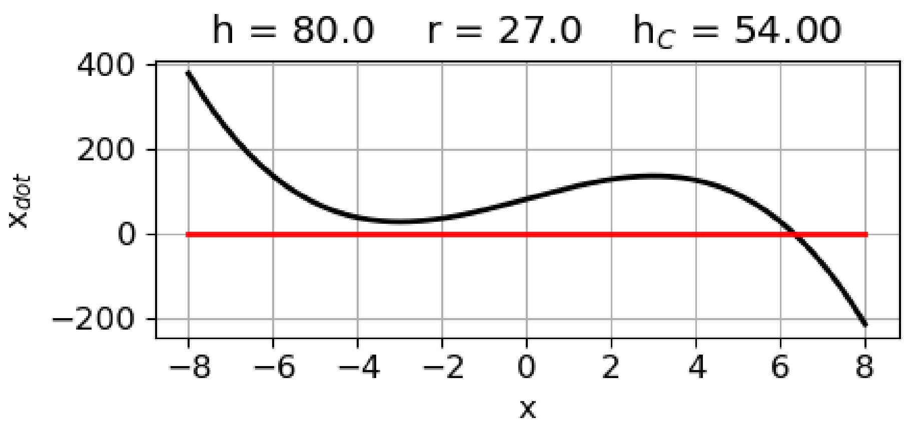


Fig. 2B. $h > h_C$

h = 80.0 r = 27.00000
Number of fixed points = 1
xe = 6.301 stable

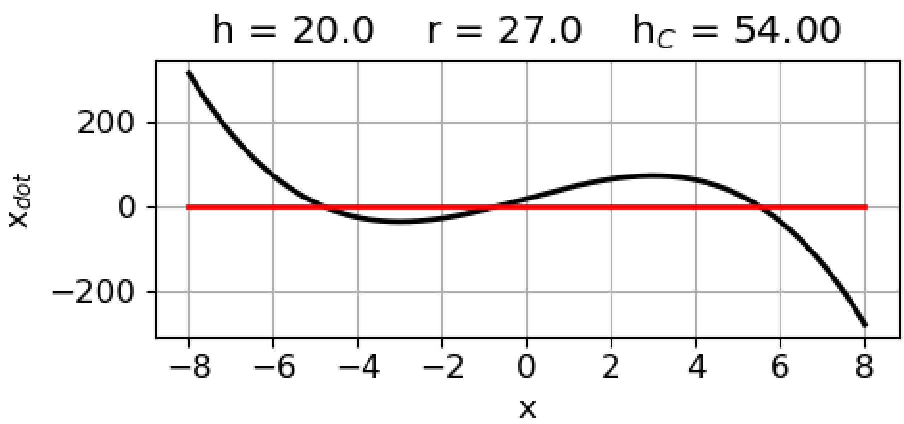


Fig. 2C. $h < h_C$

h = 20.0 r = 27.00000
Number of fixed points = 3

xe = 5.533 stable

xe = -4.776 stable

xe = -0.757 unstable

Plots in the (r, h) plane

We can draw a cusp bifurcation diagram for the critical height h_C as a function of r and h.

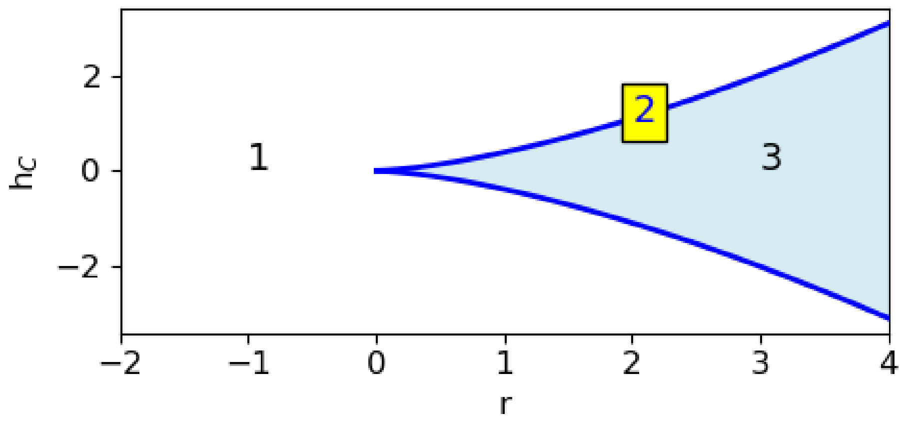


Fig 3. Parameter space cusp bifurcation diagram. The **solid blue line** gives the two fixed points for $h_C = \pm 2\left(r/3\right)^{3/2}$. The light shaded blue area is the region for 3 fixed points, and the white shaded area only 1 fixed point exists for each value of r and h. The Origin ($r = 0$, $h_C = 0$) is called the cusp point. Catastrophic events occur in the transitions between the number of fixed points as r or h change.

Bifurcation diagram x_e vs r (h = constant)

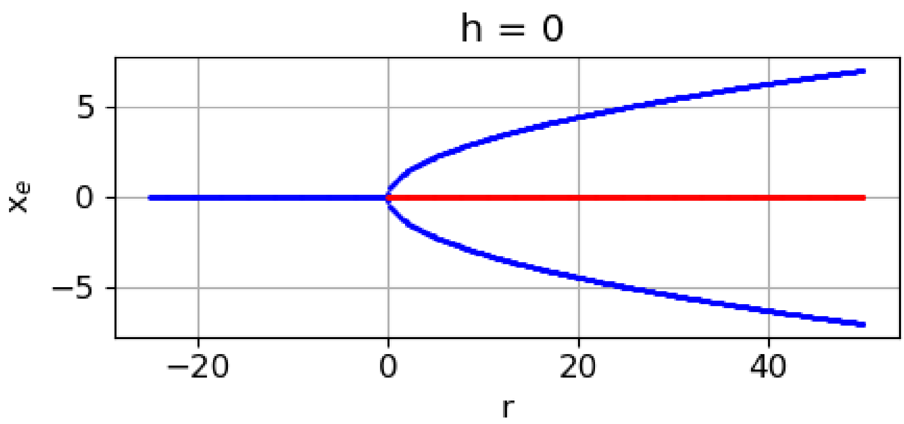


Fig. 4A. This is the standard super pitchfork bifurcation diagram when $h = 0$. The **blue** lines are the **stable fixed points** and the **unstable fixed points** are in **red**.

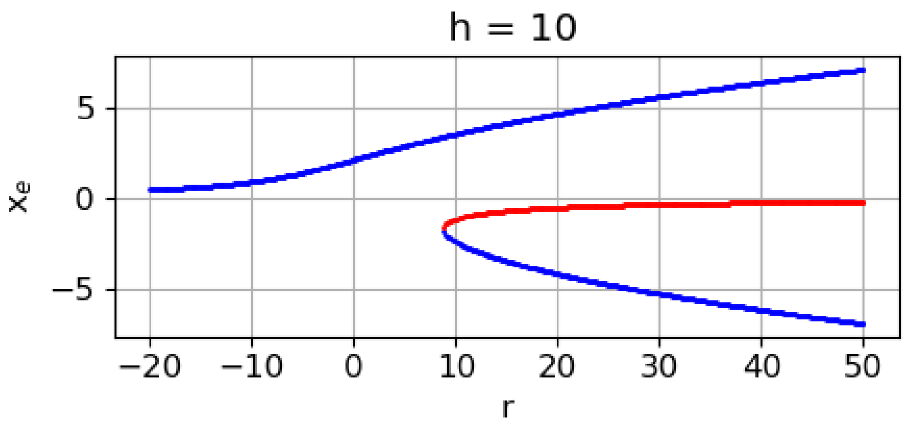


Fig. 4B. Imperfect pitchfork bifurcation

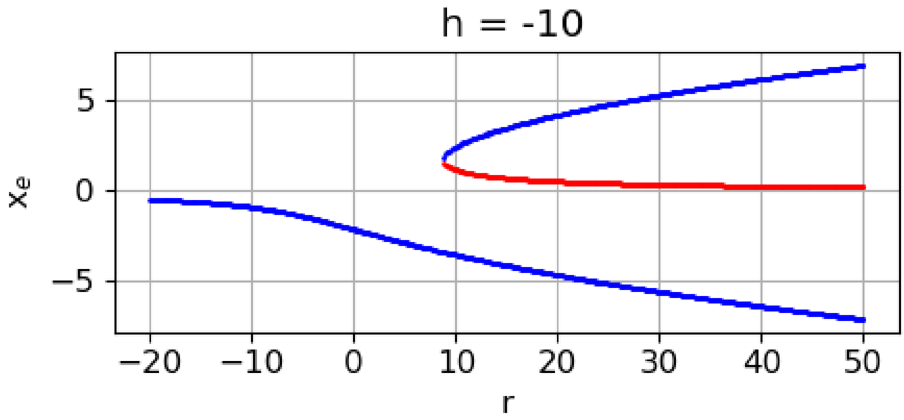


Fig. 4C. Imperfect pitchfork bifurcation

As r increases from -20 to 50, we see that we have one fixed point, then a saddle node bifurcation at the critical value of r to give two fixed points and then three fixed points. This is an example of an imperfect bifurcation diagram.

Bifurcation diagram x_e vs h (r = constant)

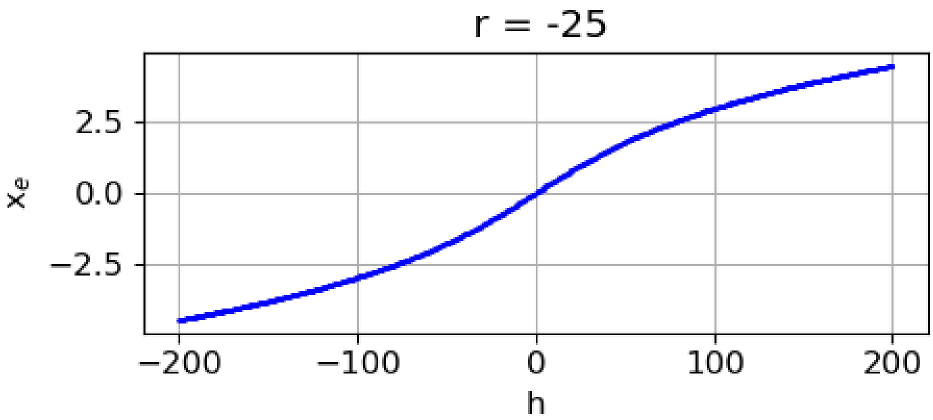


Fig. 5A. Bifurcation diagram. When $r \leq 0$ then there is only one stable fixed point for all h values.

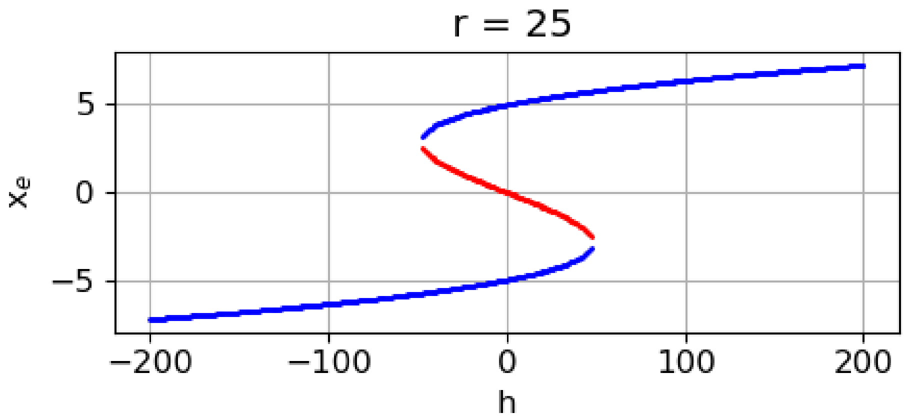


Fig. 5B. Bifurcation diagram. When $r > 0$ we get a S-shaped curve where there are one or two or three fixed points. The system is bistable since there is competition between the two fixed point stable arms. The are two saddle node bifurcation for this system.

Time evolution for the flow along the line

By comparing a bifurcation diagram with a time evolution diagram, you see that for any initial condition, you can use the bifurcation diagram to predict the flow along the time since the flow is aways directed to towards a stable fixed point and away from an unstable fixed point (the flow can be to a fixed point or plus/minus infinity). The following graphs show the case when $h = -10$.

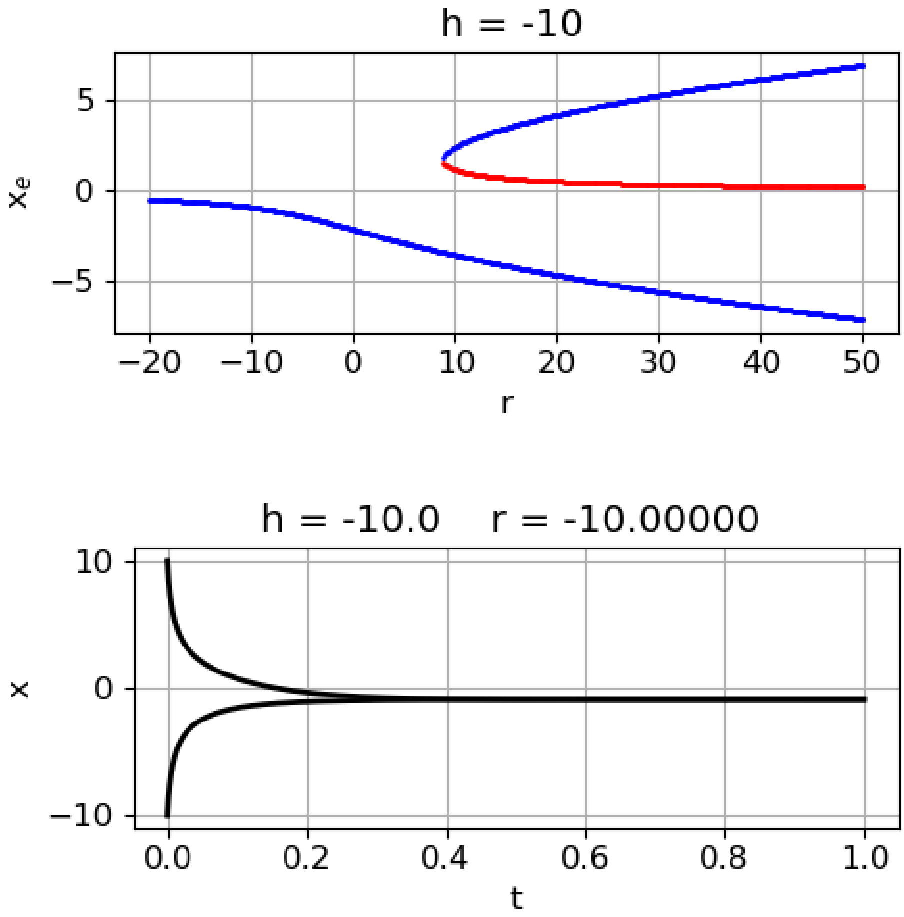
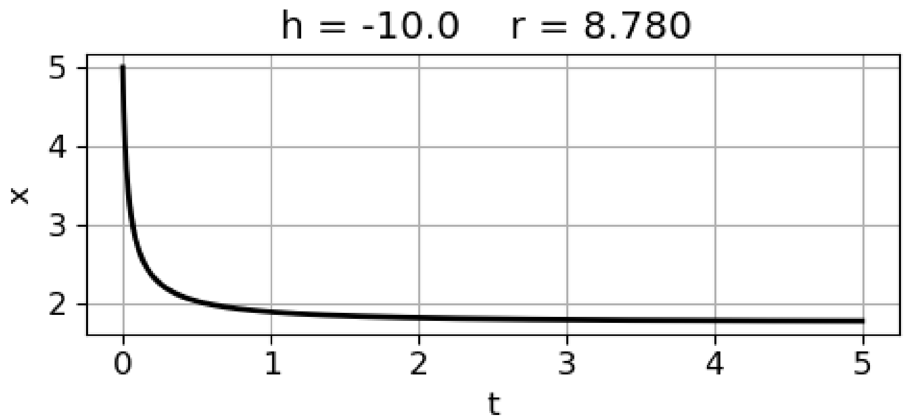
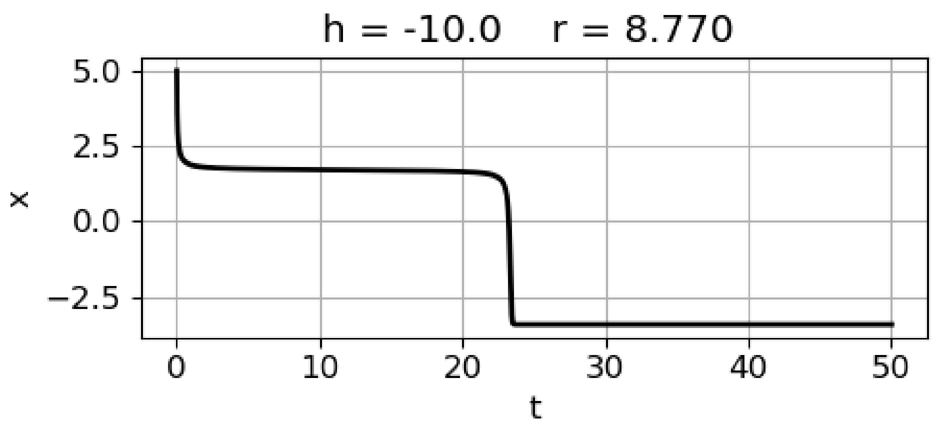


Fig. 6A. The system evolves to the single fixed point x_e = -0.922.

h = -10.0 r = -10.00000
x0 = -10.00, +10.00
xEND = -0.922
Number of fixed points = 1
xe = -0.922 stable



h = -10.0 **r = 8.78000**
cusp point: hC = -10.00 rC = 8.77205
x0 = 5.00
xEND = 1.769
Number of fixed points = 3
xe = -3.421 stable
xe = 1.762 stable
xe = 1.659 unstable



h = -10.0 **r = 8.77000**
cusp point: hC = -10.00 rC = 8.77205
x0 = 5.00
xEND = -3.420
Number of fixed points = 1
xe = -3.420 stable

Fig. 6B. Time evolution of the flow.

For an initial conditions near the cusp point $\left(h_C = -10,\ r_C = 8.77205\right)$ shows that the system response can be extremely sensitive to minute changes in a bifurcation parameter as the flow will converge to one or the other stable equilibrium point. This can make it impossible to predict the response of the system.