DOING PHYSICS WITH PYTHON

DYNAMICAL SYSTEMS

LORENZ EQUATIONS

Strange Attractors

Ian Cooper
matlabvisualphysics@gmail.com

DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive **GitHub**

ds2700.py **ds2700G.py (animation: [3D] phase portrait)**

Animation of [3D] phase portrait

Chaos is everywhere. It can crop up in unexpected places and in remarkably simple systems, and a great deal of work has been done to describe the behaviour of chaotic systems. A chaotic system is one that must show sensitivity to initial conditions, it must be topologically mixing, its orbits must be dense, and for a short time the solutions will be nearly identical to one another and as time increases the trajectories of the chaotic systems will suddenly have no correlation with the other and solutions will diverge no matter how small a change is made to the initial conditions. The idea of topological mixing implies that the system will evolve over time such that every open set of its phase space will eventually intersect with every other open region. The density of the orbits is also of importance to prove that a system is acting chaotically. Orbits may never come close to anything resembling repeating themselves.

As an example, we will consider the Lorenz System. This system is one of the earlier examples of chaotic behaviour and was discovered by Edward Lorenz in 1963, while working the dynamics of the atmosphere. The Lorenz system is described by the set of ODEs

$$\text{(1)} \qquad \begin{aligned} \dot{x} &= -s(x-y) \\ \dot{y} &= r\,x - y - x\,z \\ \dot{z} &= x\,y - b\,z \end{aligned}$$

where *s*, *r*, *b* are the system constants and *x*, *y*, *z* are the state variables. The *r* parameter will be considered as the bifurcation parameter and the system response will be investigated as the value of *r* is varied.

s > 0 is the Prandtl number and it represents the ratio of fluid viscosity to thermal conductivity (ratio of how quickly fluid flows through the system to how effective it absorbs heat in contact with other molecules).

r > 0 is the normalized Rayleigh number and it represents the difference in temperature between the top and bottom of the atmospheric column.

b > 0 depends upon the geometry of the domain and simply describes the bounds of the system.

The most popular values for the parameters are:

$$s = 10 \quad r = 28 \quad b = 8/3$$

Ed Lorenz derived this three-dimensional system from a drastically simplified model of convection rolls in the atmosphere. Convection rolls are organized, counter-rotating rolls of air that form in the atmospheric boundary layer. They are formed when the ground is heated, causing air to rise in updrafts and sink in downdrafts, and are often visible from the ground as long lines of cumulus clouds parallel to the low-level wind. These rolls play a significant role in transporting heat, moisture, and momentum in the atmosphere.

Fig. 1. Convection rolls.

Lorenz discovered that this simple-looking deterministic system could have extremely erratic dynamics over a wide range of parameters. The solutions oscillate irregularly, never exactly repeating but always remaining in a bounded region of phase space. When he plotted the trajectories in three dimensions, he discovered that they settled onto a complicated set, now called a strange attractor. Unlike stable fixed points and limit cycles, the strange attractor is not a point or a curve or even a surface—it's a fractal, with a fractional dimension between 2 and 3. In this article, we'll follow the beautiful chain of reasoning that led Lorenz to his discoveries with the goal to get a feel for his strange attractor and the chaotic motion that occurs on it.

Lorenz showed that in a certain range of parameters, there could be no stable fixed points and no stable limit cycles, but showed that all trajectories remain confined to a bounded region and are eventually attracted to a set of zero volume where trajectories move on a strange attractor with chaotic motion in phase space.

The Lorenz system is dissipative: volumes in phase space contract under the flow exponentially fast. If we start with an enormous solid blob of initial conditions, it eventually shrinks to a limiting set of zero volume, like a balloon with the air being sucked out of it. All trajectories starting in the blob end up somewhere in this limiting set which consists of fixed points, limit cycles, or for some parameter values a strange attractor. Volume contraction imposes strong constraints on the possible solutions of the Lorenz equations. The mathematics of showing this concept is not done.

It is impossible for the Lorenz system to have either repelling fixed points or repelling closed orbits (all trajectories starting near the fixed point or closed orbit are driven away from it). This is because repellers are incompatible with volume contraction because they are sources of volume - suppose we encase a repeller with a closed surface of initial conditions nearby in phase space by considering a small sphere around a fixed point, or a thin tube around a closed orbit. A short time later, the surface will have expanded as the corresponding trajectories are driven away. Thus, the volume inside the surface would increase. This contradicts the fact that all volumes contract. By a process of elimination, we conclude that all fixed points must be sinks or saddles, and closed orbits (if they exist) must be stable or saddle-like.

Fixed points

The Lorenz system given by equation 1 has two types of fixed points (x_e, y_e, z_e). The Origin (0, 0, 0) is a fixed point for all values of the parameters. If $r \leq 1$ then the only fixed point is the Origin (0 ,0, 0) and the Origin is a global attractor and the motion freezes at the Origin.

For $r > 1$, there is also an additional symmetric pair of fixed points known as C^+ and C^- because of the symmetry

$x \to -x,\ y \to -y,\ z \to -z \Rightarrow$ gives the same system equations

In other words, because of the symmetry, all solutions are either symmetric themselves, or have a symmetric partner.

$r > 1$

x nullcline $\quad \dot{x} = 0 \quad \Rightarrow \quad y_e = x_e$

y nullcline $\quad \dot{y} = 0 \quad \Rightarrow \quad z_e = r - 1$

z nullcline $\quad \dot{z} = 0 \quad \Rightarrow \quad z_e = x_e^{\,2} / b \Rightarrow x_e = \pm\sqrt{b(r-1)}$

$$C^+ \quad x_e = +\sqrt{b(r-1)} \quad y_e = +\sqrt{b(r-1)} \quad z_e = r-1$$
$$C^- \quad x_e = -\sqrt{b(r-1)} \quad y_e = -\sqrt{b(r-1)} \quad z_e = r-1$$

They represent steady state left or right turning convection rolls (figure 1). As $r \to 1^+$, C^+ and C^- coalesce with the Origin in a pitchfork bifurcation.

There is a change in the number of fixed points as r increases through $r = 1$. So, $r = 1$ is a bifurcation point because of the change in the nature of the solutions $x(t)$, $y(t)$ and $z(t)$.

Stability of the fixed points

Finding the eigenvalues of the Jacobian matrix **J** can help determine the stability of the fixed points (x_e, y_e, z_e).

$$\mathbf{J}(x_e, y_e, z_e) = \begin{pmatrix} \partial f / \partial x & \partial f / \partial y & \partial f / \partial z \\ \partial g / \partial x & \partial g / \partial y & \partial g / \partial z \\ \partial h / \partial x & \partial h / \partial y & \partial h / \partial z \end{pmatrix}_{(x_e, y_e, z_e)}$$

$$\mathbf{J}(x_e, y_e, z_e) = \begin{pmatrix} -s & s & 0 \\ r - z_e & -1 & -x_e \\ y_e & x_e & -b \end{pmatrix}$$

For the fixed point at the Origin (0, 0, 0)

$$\mathbf{J}(0,0,0) = \begin{pmatrix} -s & s & 0 \\ r & 0 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

The eigenvalues of the Jacobian **J** are found using the Python function **eig**. Figure 2 shows the values of the three eigenvalues as a function of the bifurcation parameter r for the fixed point at the Origin (0,0,0).

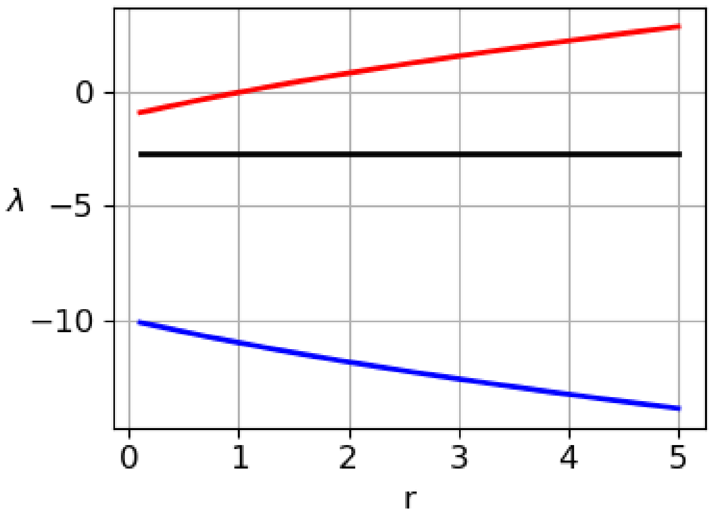


Fig. 2. The three eigenvalues of the Jacobian **J** as a function of the bifurcation parameter for the fixed point at the Origin (0,0,0).

For r < 1 all three eigenvalues are negative. All directions are incoming and the Origin is a stable node (sink). However, when $r > 1$, we have two negative eigenvalues and one positive eigenvalue. We now have two incoming directions and one outgoing direction. So, the Origin is now a saddle node. As r crosses $r = 1$ we have a pitchfork bifurcation as the Origin changes its stability from stable to saddle and two new fixed points C^+ and C^- are created.

For $r < 1$, you can show that every trajectory approaches the Origin as $t \to \infty$, thus the Origin is globally stable and there can be no limit cycles or chaos (figure 3).

$$t \to \infty \quad x(t) \to 0,\ y(t) \to 0,\ z(t) \to 0$$

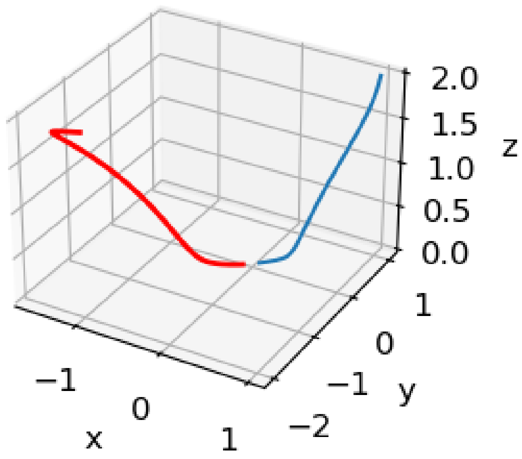


Fig. 2A. $r = 0.8 < 1$. The trajectories are attracted to the stable fixed point, the Origin (0 ,0, 0). The plot clearly shows the contraction in the phase space volume as all trajectories are pulled together at the Origin.

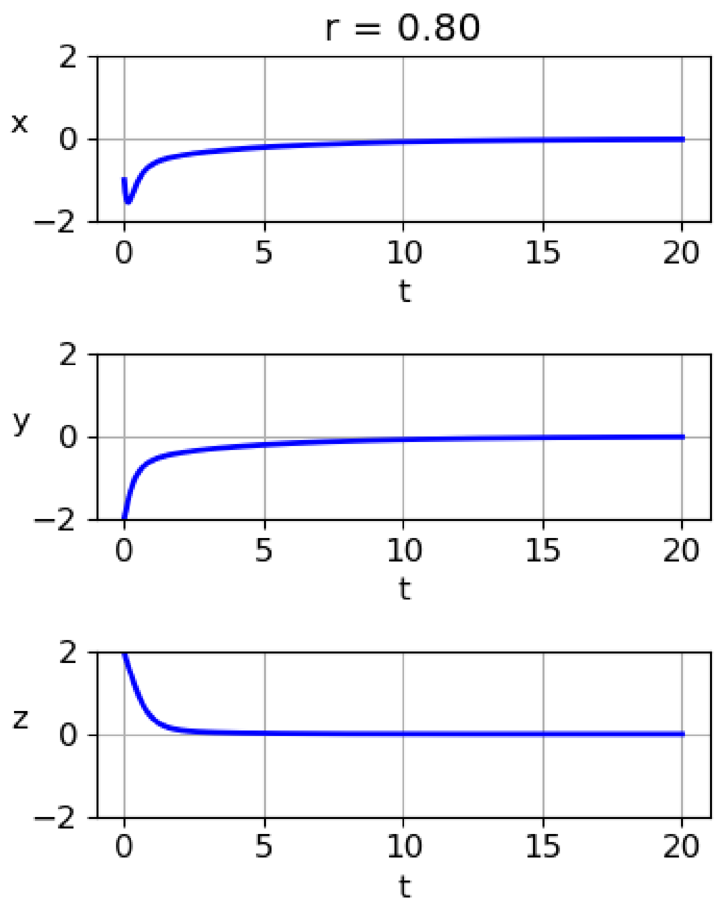


Fig. 2B. Time evolution of the state variables for $r = 0.8 < 1$.

Stability of C^+ and C^-

For $r > 1$

$$C^+ \quad x_e = +\sqrt{b(r-1)} \quad y_e = +\sqrt{b(r-1)} \quad z_e = r-1$$

$$C^- \quad x_e = -\sqrt{b(r-1)} \quad y_e = -\sqrt{b(r-1)} \quad z_e = r-1$$

Figure 3 shows the fixed points other than the Origin when $r > 1$.

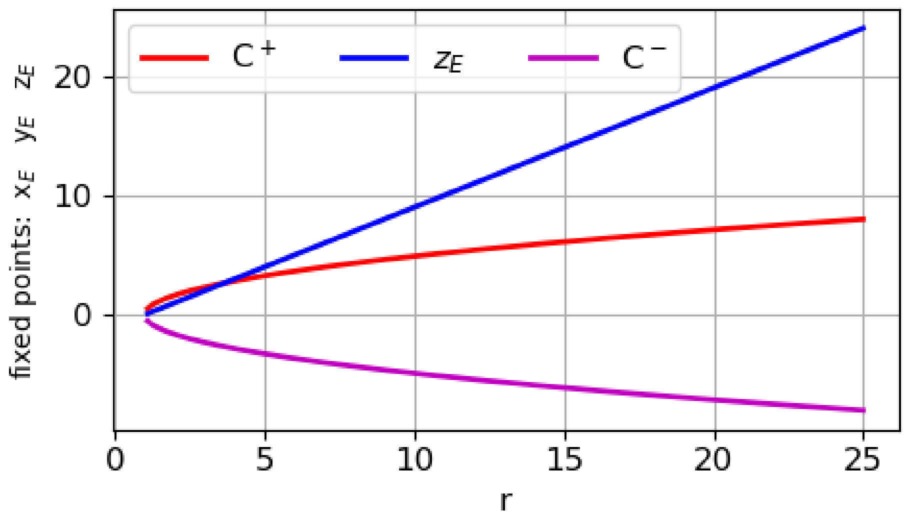


Fig. 3. Variation in the fixed points C^+ (x_{E1}, y_{E1}) and C^- (x_{E2}, y_{E2}) as a function of the bifurcation parameter *r*.

For the stability of the two fixed points C^+ and C^-, the eigenvalues as a function of *r* are computed from the Jacobian using the Python function **eig**. The results of the computation are shown in figure 4. The two fixed points have identical eigenvalues.

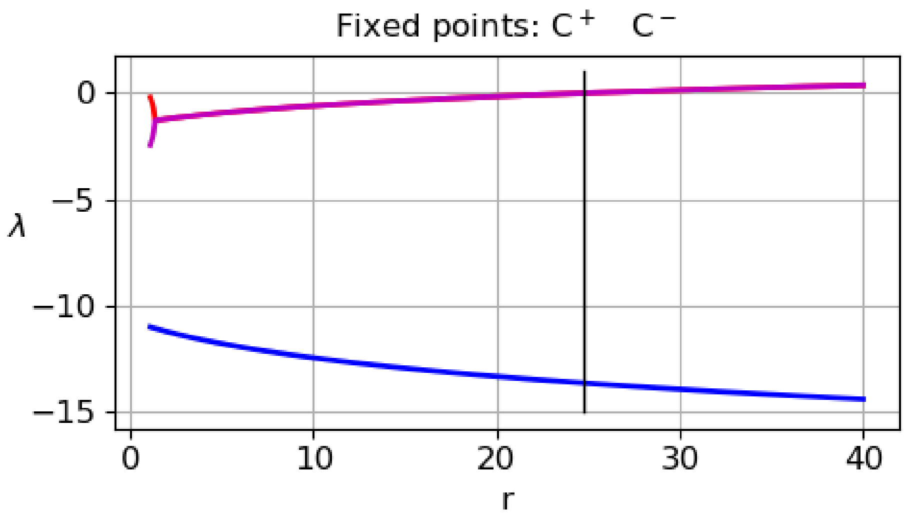


Fig. 4. The three eigenvalues for the fixed points C^+ and C^-.

From figure 4, you see that the two fixed points C^+ and C^- are stable when $1 < r < r_H$ (black vertical line, r_H = **24.76**) since all three eigenvalues are negative. But, when $r > r_H$, two of the eigenvalues become positive and the fixed points lose their stability. That is, C^+ and C^- lose stability in a subcritical Hopf bifurcation at $r = r_H$.

A linear analytical for the stability gives

$$1 < r < r_H \qquad r_H = \frac{s\,(s+b+1)}{s-b-1} = 24.74$$

So, our numerical estimate for r_H is in excellent agreement with the analytical prediction.

The fixed points C^+ and C^- are stable when $r < r_H$. However, the limit cycles are unstable and exist only for $1 < r < r_H$. The stable fixed points are encircled by saddle cycles, a new type of unstable limit cycle that is possible only in phase spaces of three or more dimensions.

As $r \to r_H$ from below, the cycle shrinks down around the fixed point. (figure 5). At the Hopf bifurcation, the fixed point absorbs the saddle cycle and changes into a saddle point. For $r > r_H$ there are no attractors in the neighbourhood. So, for $r > r_H$ trajectories must fly away to a distant attractor. But what can it be? Could it be that all trajectories are repelled out to infinity? No! we can prove that all trajectories eventually enter and remain in a certain large ellipsoid. So, the trajectories must have a bizarre kind of long-term behaviour, like balls in a pinball machine, they are repelled from one unstable object after another. At the same time, they are confined to a bounded set of zero volume, yet they manage to move on this set forever without intersecting themselves or others. We can get out of this conundrum as a result of strange attractors and chaos.

Figure 5 shows two trajectories with different initial conditions. Since $1 < r = 20 < r_H = 24.8$ the two fixed points C^+ and C^- are stable and the Origin is unstable. Depending upon the initial condition, a trajectory will spiral inwards towards either C^+ or C^-.

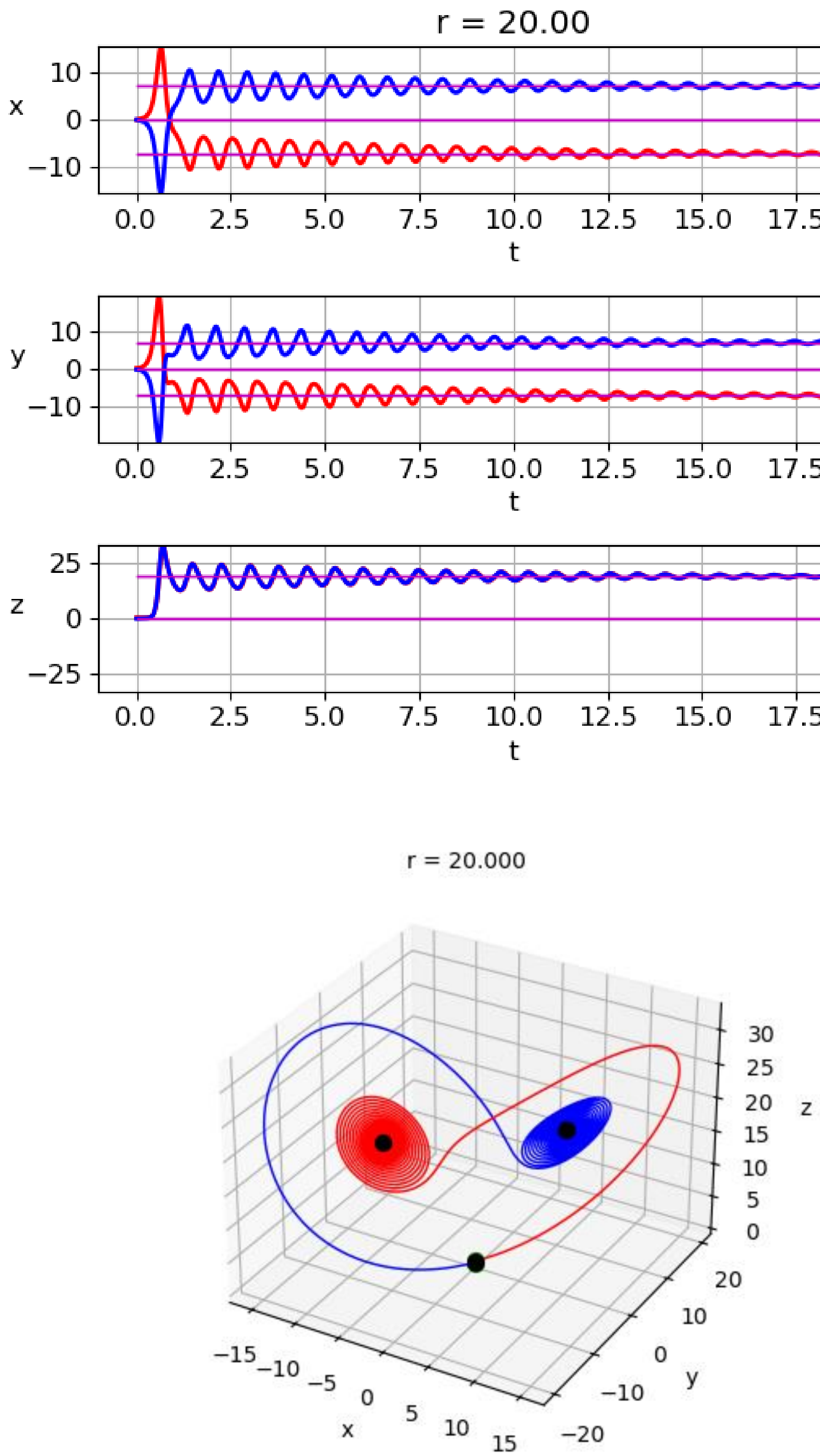


Fig. 5. $1 < r < r_H = 24.74$. The **blue** and **red** trajectories (different initial conditions) are pulled to different fixed points C^+ or C^- and repelled from the Origin.

Bifurcation parameter r = 20.000

fixed points: xE = 7.118 yE = 7.118 zE = 19.000

xE = -7.118 yE = -7.118 zE = 19.000

Eigenvalues:

lambda1 = -13.357 + 0.000 j

lambda2 = -0.155 + 8.709 j

lambda3 = -0.155 + -8.709 j

Simulation time tS = 20.00

Initial conditions: x(0) = 0.000 y(0) = -0.200 z(0) = -0.200

End points: xF = 6.911 yF = 6.810 zF = 18.932

Initial conditions: x(0) = 0.000 y(0) = 0.200 z(0) = 0.200

End points: xF = -6.928 yF = -6.794 zF = 18.999

All the real parts of the three eigenvalues are negative, and so the fixed points are stable.

A very different behaviour of the Lorenz system occurs when $r > r_H = 24.74$. An as example the case where $r = 28$ is considered.

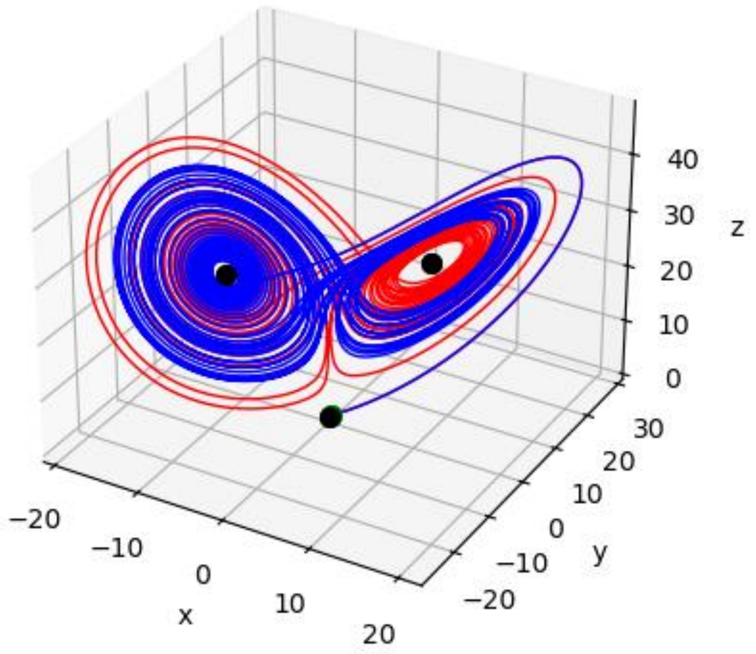


Fig. 6A. r = 28. The initial conditions are almost identical, the difference being x(0) = 0.20 and x(0) = 0.21.

The trajectories for $r > r_H$ trace out a strange attractor and the motion is chaotic. Although the motion is deterministic, it is not predictable. It is impossible to predict with certainty the trajectory for long time periods since we never know with one hundred percent certainty the initial conditions as shown in figure 7. Thus, after an initial transient, the solution settles into an irregular oscillation that persists as $t \to \infty$, but never repeats exactly. The motion is aperiodic. Figure 6A shows the wonderful butterfly structure that emerges in the phase space plot of the trajectory in phase space where there no self-intersections occur.

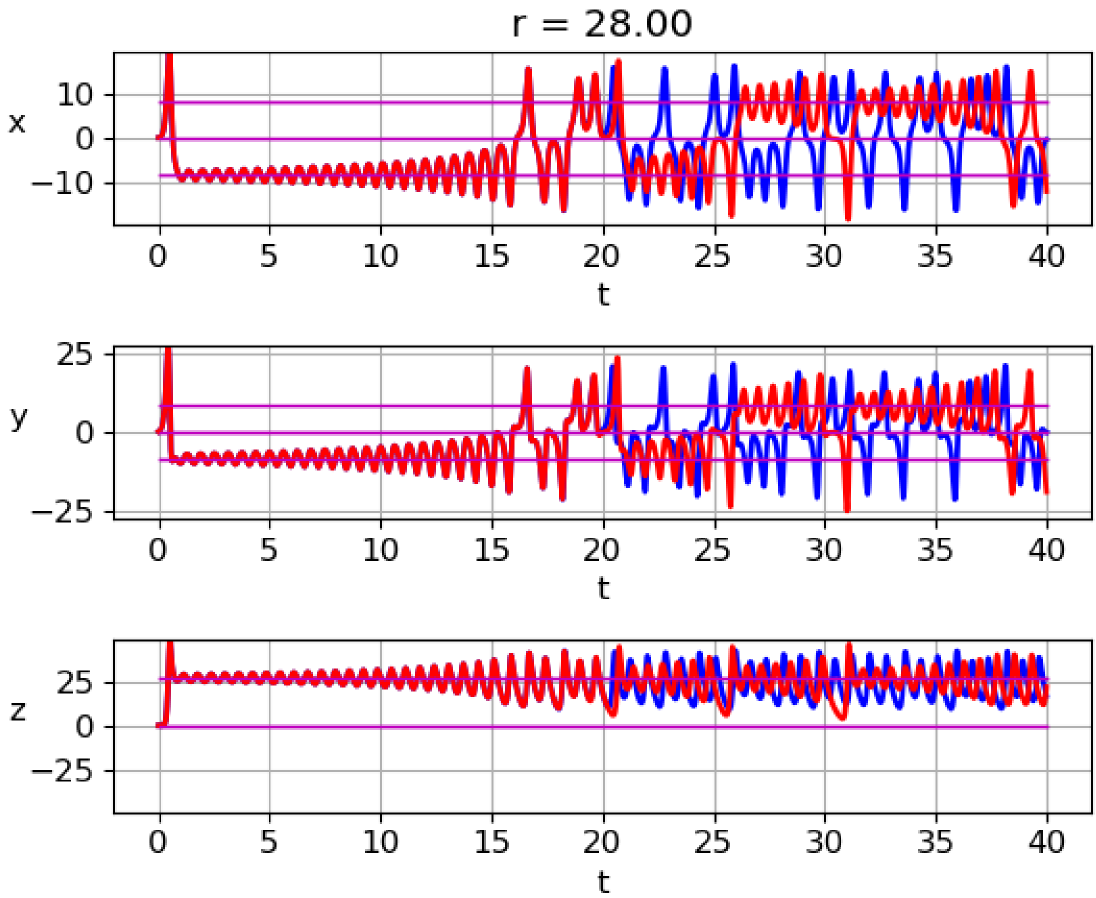


Fig. 6B. The initial conditions are almost identical, the difference being x(0) = 0.20 and x(0) = 0.21. For $t < 18$ the trajectories are almost identical, but for $t > 18$ they become very different.

The trajectory starts near the initial position, then swings to the right, and then dives into the centre of a spiral on the left. After a very slow spiral outward, the trajectory shoots back over to the right side, spirals around a few times, shoots over to the left, spirals around, and so on indefinitely. The number of circuits made on either side varies unpredictably from one cycle to the next. In fact, the sequence of the number of circuits has many of the characteristics of a random sequence. The trajectory appears to settle onto an exquisitely thin set that looks like a pair of butterfly wings. This limiting set is the attracting set of zero volume.

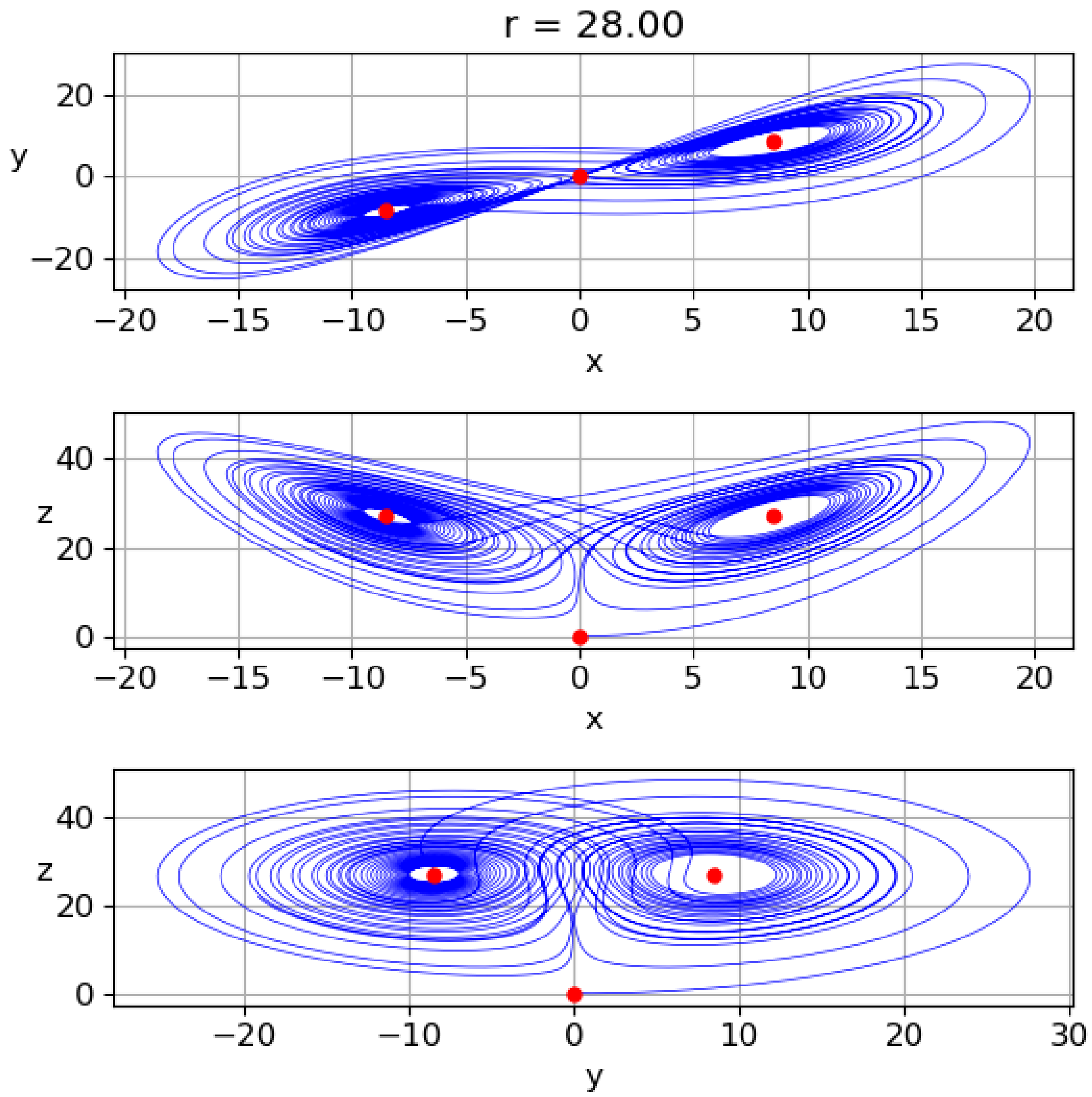


Fig. 6C. [2D] phase portraits.

Figure 7 shows a bifurcation diagram with r as the bifurcation parameter. The magenta curves are for the fixed points x_e and the red and blue dots are the final trajectory positions x_F.

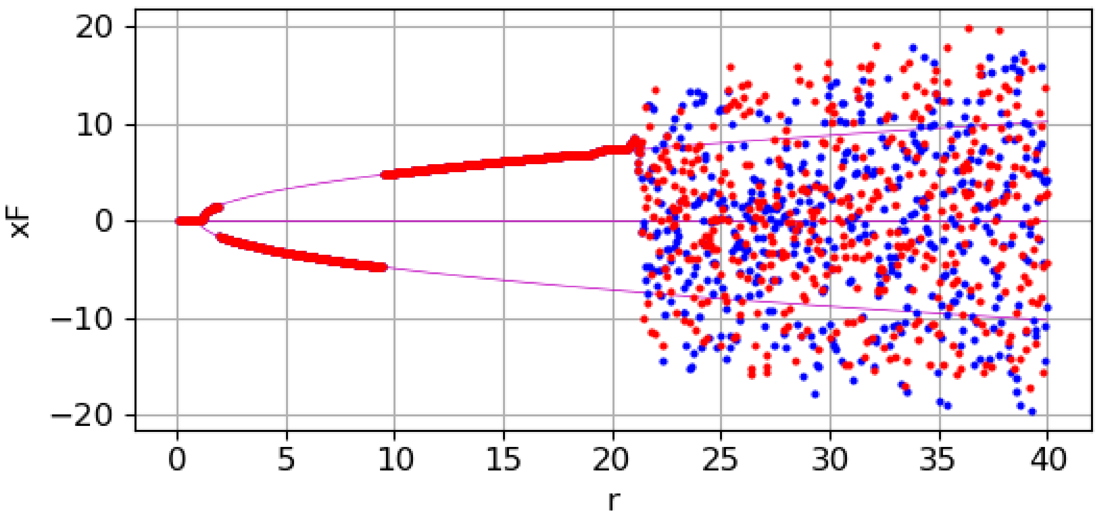
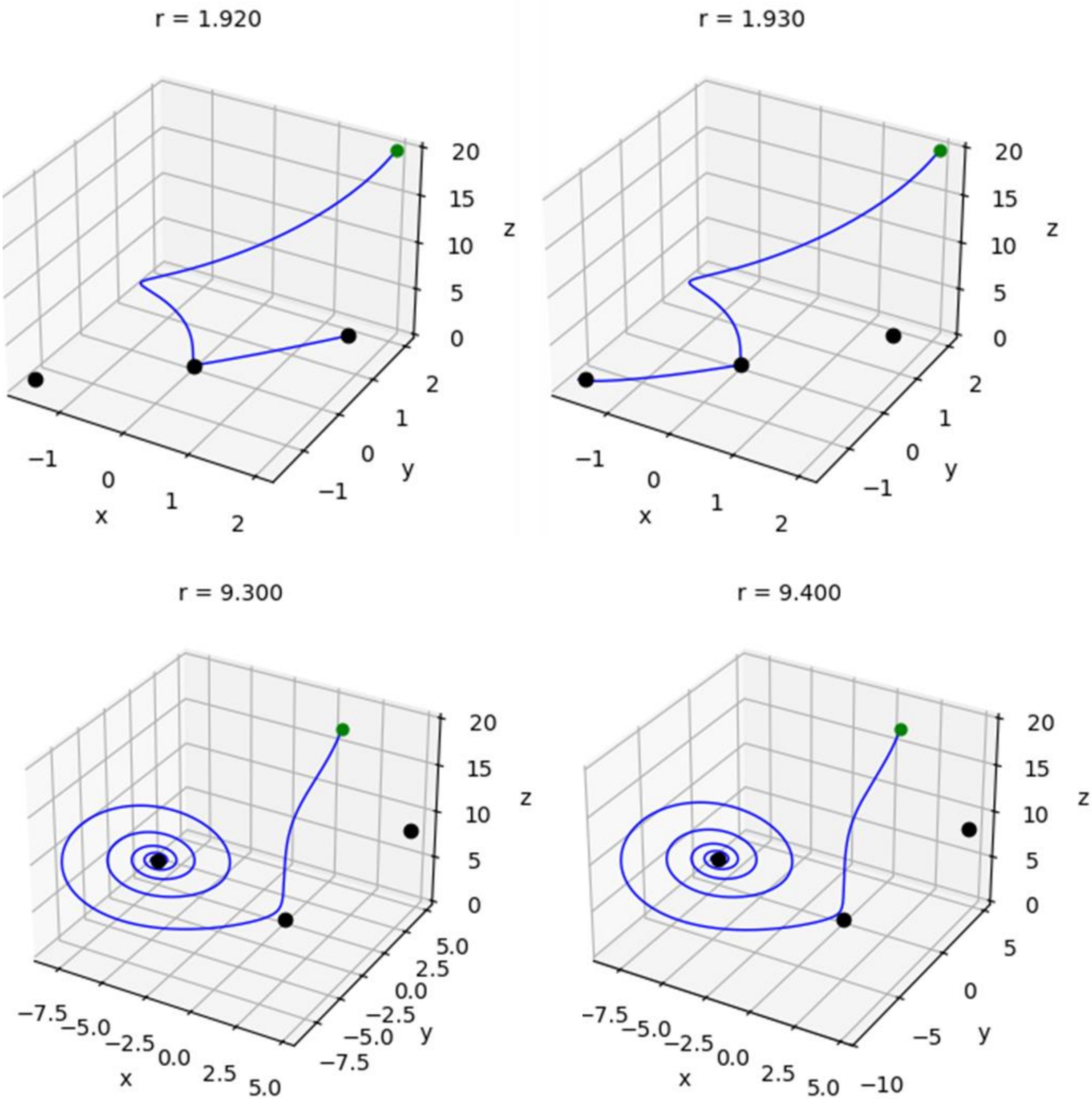


Fig. 7. The initial conditions are (**2.000**, 2, 20) and (**2.001**,2,20). The magenta lines are for the three fixed points (0,0,0), C^+ and C^-.

For $r < 1$, the Origin (0, 0, 0) is the only fixed point and is globally stable. A supercritical pitchfork bifurcation occurs at $r = 1$.
The Origin (0, 0, 0) changes its stability and a symmetric pair of attracting (stable) fixed points C^+ and C^- are created.

For $1 < r < r_H$ both trajectories are almost identical since they have almost identical initial $x(0)$ values and for $r < 1.9$ they converge to the fixed point C^+. But at $r = 1.9$, there is a transition $C^+ \to C^-$ and then at $r = 9.5$ the reverse transition occurs $C^- \to C^+$.


r = 1.920
r = 1.930
r = 9.300
r = 9.400
x
y
z

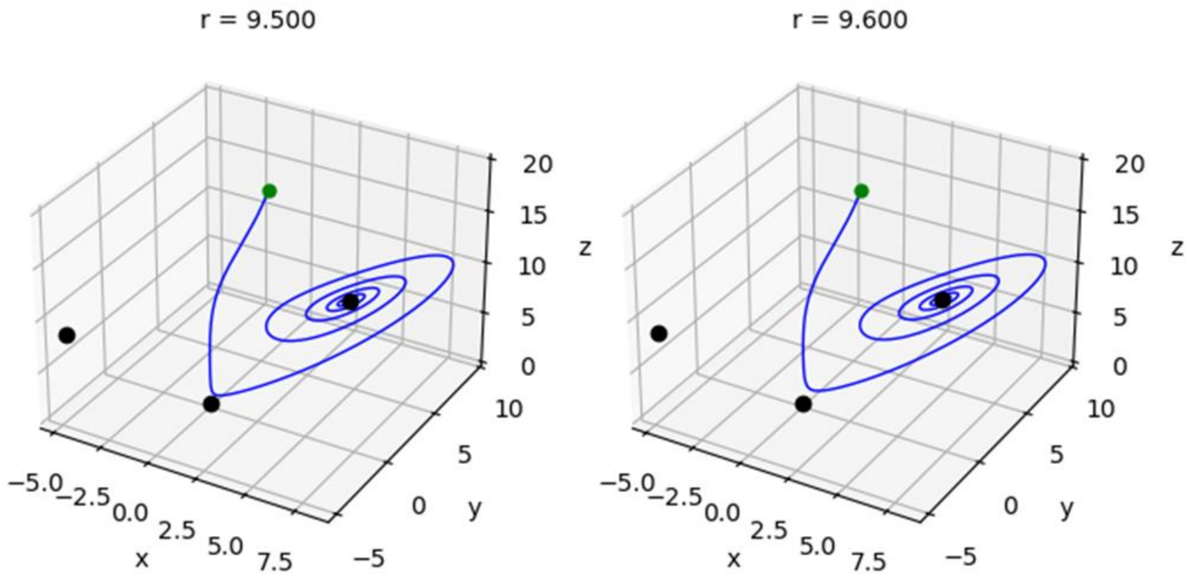


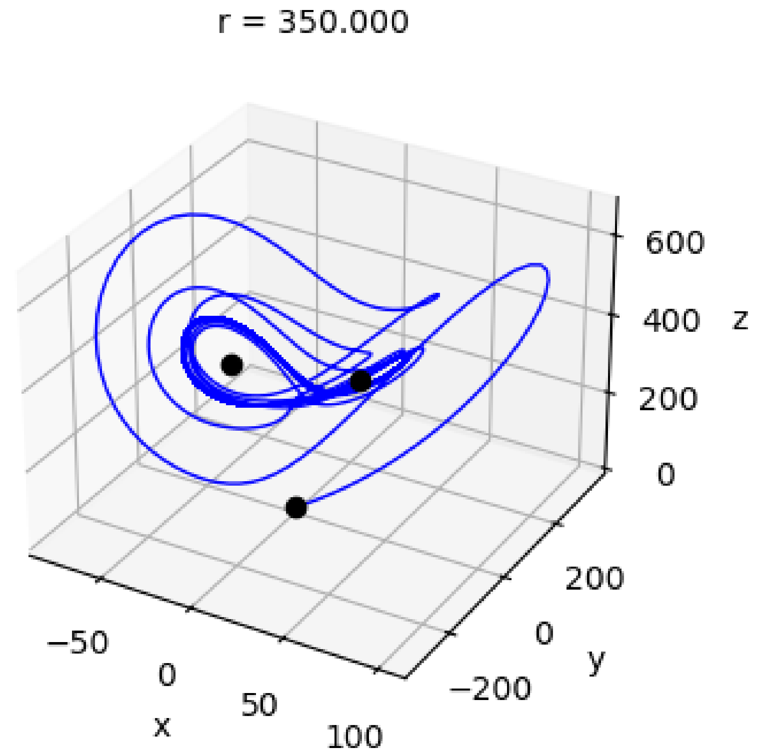
Fig. 8. The initial conditions are (2.00, 2.00, 20.0).

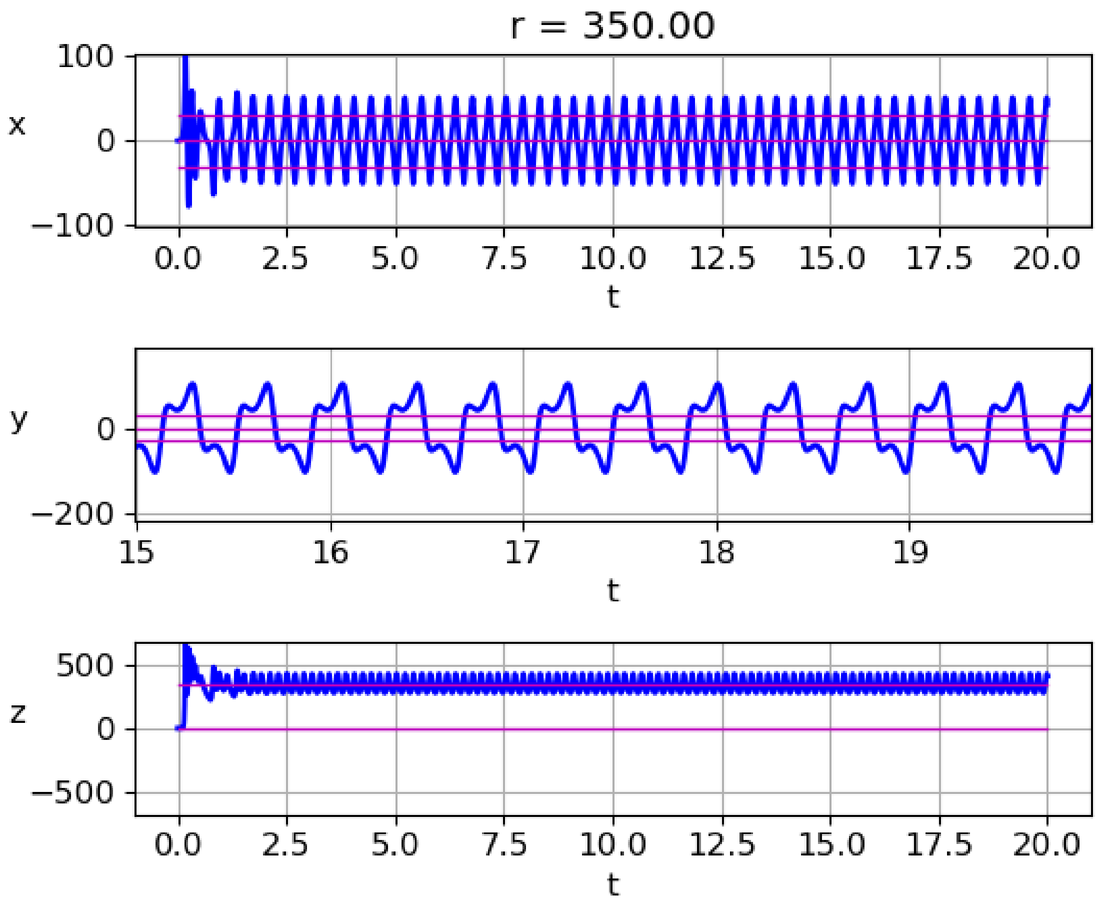
A homoclinic orbit lies in the intersection of the stable manifold and the unstable manifold of an equilibrium. A homoclinic bifurcation occurs when a limit cycle collides with a saddle point, producing complicated, and possibly chaotic dynamics.

For the initial conditions (2.00, 2.00, 20.0) for the trajectories shown in figure 8, we see that when $r = 1.92$ the trajectory approaches close to the saddle node at the Origin and terminates at the C^+ fixed point. When r is increment to $r = 1.93$, the trajectory is now repelled from the saddle and the trajectory goes to the C^- fixed point. Further incrementing the r value leads to decaying orbits circulating around the C^- fixed point. When $r = 9.3$ and $r = 9.4$ the circulating trajectory gets closer to the saddle node at the Origin. At $r = 9.5$ the unstable limit cycle has grown such that it intersects with the saddle point at the Origin leading to the trajectory now being attracted to the C^+ fixed point. This is called a homoclinic orbit. Hence, we have two homoclinic bifurcations occurring at $r = 1.93$ and 9.5. Trajectories rattle around chaotically for a while, but eventually escape and settle down to C^+ or C^-. We get transient chaos and it shows that a deterministic system can be unpredictable, even if its final states are very simple

At $r_H = 24.74$ the fixed points lose stability by absorbing an unstable limit cycle in a subcritical Hopf bifurcation. For $r > r_H = 24.7$, there are three fixed points (0,0,0), C^+ and C^-. The trajectories fly off to a strange attractor. A strange attractor is a set of states in a chaotic dynamical system that is globally stable but locally unstable, meaning it attracts points toward it but is also highly sensitive to initial conditions, causing nearby points to diverge exponentially over time. The system's behaviour is unpredictable in the short term but statistically predictable in the long term, exhibiting a fractal structure and often visualized as a complex, geometric shape like the famous Lorenz attractor shown in figure 6.

Numerical simulations indicate that the system has a globally attracting limit cycle for all $r > 313$. In figure 9 a plot of the solution for $r = 350$ is shown where the trajectory approaches a limit cycle.


r = 350.000
600
400
200
0
z
200
0
−200
y
−50
0
50
100
x


r = 350.00
100
0
−100
x
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
t
y
0
−200
15
16
17
18
19
t
500
0
−500
z
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
t

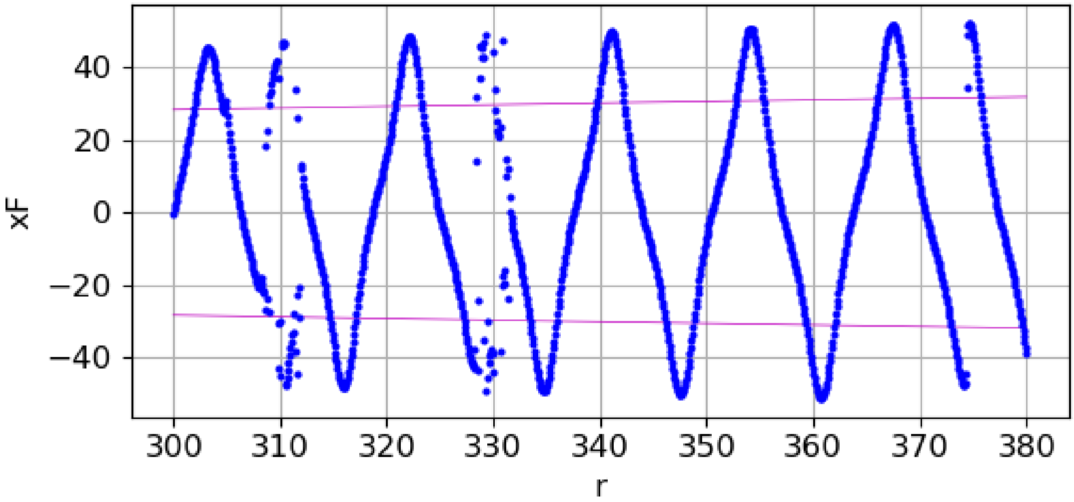


Fig. 9. Solution of high values of the bifurcation parameter.
For $r = 350$, the flow of the system is aperiodic and not chaotic.

The story becomes more complicated for $28 < r < 313$. For most values of r one finds chaos, but there are also small windows of periodic behaviour interspersed. This alternating pattern of chaotic and periodic regimes resembles that seen in the logistic map.
A large window exists in the range are $98 < r < 102$ as shown in figure 10.

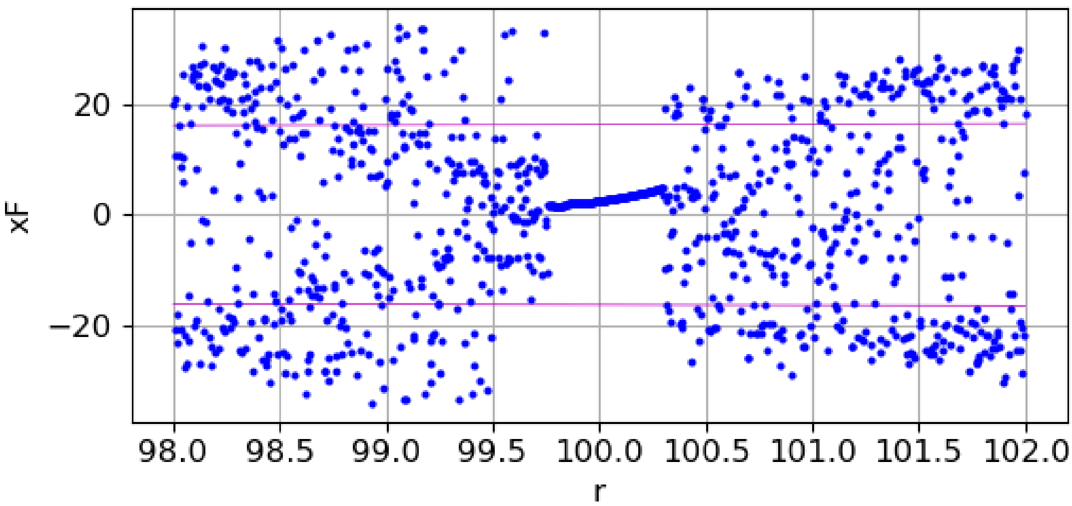


Fig. 10. Bifurcation diagram for $r = 98$ to $r = 102$. The flow is not chaotic in the range $99.83 < r < 100.3$.

Defining Chaos

No definition of the term chaos is universally accepted yet, but almost everyone would agree on:

Chaos is aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on initial conditions.

Aperiodic long-term behaviour means that there are trajectories which do not settle down to fixed points, periodic orbits, or quasiperiodic orbits as $t \to \infty$.

Deterministic means that the system has no random or noisy inputs or parameters. The irregular behaviour arises from the system's nonlinearity, rather than from noisy driving forces.

Sensitive dependence on initial conditions means that nearby trajectories separate exponentially fast, i.e., the system has a positive Lyapunov exponent.

REFERENCES

Jason Bramburger

The Lorenz Equations - Dynamical Systems | Lecture 27

https://www.youtube.com/watch?v=0Rpy-xSsAo8&t=1622s

Steven H. Strogatz

Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering

https://www.youtube.com/watch?v=Q_f1vRLAENA

https://ocw.mit.edu/courses/res-18-009-learn-differential-equations-up-close-with-gilbert-strang-and-cleve-moler-fall-2015/

Lyapunov exponents

https://github.com/Ceyron/machine-learning-and-simulation/blob/main/english/simulation_scripts/lorenz_lyapunov_exponent_numpy.ipynb

https://github.com/Ceyron/machine-learning-and-simulation/blob/main/english/simulation_scripts/lorenz_lyapunov_spectrum_jax.ipynb