DOING PHYSICS WITH PYTHON

COMPLEX SYSTEMS

[2D] LINEAR DYNAMICAL SYSTEMS

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cs200.py

This Python Code solves a pair of linear ODEs in x and y. The solution gives the time evolution of the two variables and the phase portrait (quiver plot and streamplot) using nullclines, vector fields, and eigenvectors. One can find and classify critical points in the phase plane.

INTRODUCTION

This article considers how Python can be used to solve [2D] linear dynamical systems. The [2D] systems are described by a pair of ordinary differential equations (ODEs) in *x* and *y*. The ODEs are solved numerically using the Python function **odeint**. The solutions for *x* and *y* are displayed graphically as time evolution plots and phase portrait plots. For a linear system, the ODEs can be expressed as

$$\dot{x} = a_{11}\,x + a_{12}\,y \qquad \dot{y} = a_{21}\,x + a_{22}\,y$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

The phase portrait is a [2D] figure showing how the qualitative behaviour of system is determined as *x* and *y* vary with *t*. With the appropriate number of trajectories plotted, it should be possible to determine where any trajectory will end up from any given initial condition.

The direction field or **vector field** gives the gradients *dy* *dx*. The slope of the trajectories at each point in the vector field is given by

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$$

The contour lines for which *dy/dx* is a constant are called **isoclines**. The contour lines for which *dy/dt* = 0 and *d/xdt* = 0 are called ***nullclines***.

Isoclines may be used to help with the construction of the phase portrait. For example, the nullclines for which $\dot{x} = 0$ and $\dot{y} = 0$ are used to determine where the trajectories have vertical and horizontal tangent lines, respectively. If $\dot{x} = 0$, then there is no motion horizontally, and trajectories are either stationary or move vertically. When $\dot{y} = 0$, then there is no motion vertically, and trajectories are either stationary or move horizontally.

A linear system is **nonsimple** if the matrix **A** is singular, i.e., det(**A**) = 0, and at least one of the eigenvalues is zero. The system then has critical points other than the Origin. Systems in which $\det(\mathbf{A}) \neq 0$ is **simple**, and the Origin is then the only critical point.

An **equilibrium** occurs at **critical points** (**fixed points**) of a dynamical system generated by system of ordinary differential equations (ODEs) where a solution that does not change with time. The stability of typical equilibria of smooth ODEs is determined by the sign of real part of eigenvalues of the Jacobian matrix. These eigenvalues are often referred to as the eigenvalues of the equilibrium. In [2D] systems the Jacobian matrix is

$$\mathbf{J}(x, y) = \begin{pmatrix} \partial f / \partial x & \partial f / \partial y \\ \partial g / \partial x & \partial g / \partial y \end{pmatrix}$$

and has two eigenvalues, which are either both real or complex-conjugate. The eigenvalues and eigenfunctions can be found using the Python function **eig**.

Eigenvalues λ_1, λ_2	Stability of critical point (equilibrium or fixed points)
distinct, real, and positive $\lambda_1 \neq \lambda_2 \quad \lambda_1 > 0 \quad \lambda_2 > 0$	Unstable node
Both eigenvalues zero $\lambda_1 = \lambda_2 = 0$	Unstable
distinct, real, and negative $\lambda_1 \neq \lambda_2 \quad \lambda_1 < 0 \quad \lambda_2 < 0$	Stable node
One eigenvalue is positive and the other negative $\lambda_1 > 0 \quad \lambda_2 < 0$	Saddle point
Repeated eigenvalues $\lambda_1 = \lambda_2 > 0$	Unstable If there are two linearly independent eigenvectors, then the critical point is called a singular node. If there is one linearly independent eigenvector, then the critical point is called a degenerate node.
Repeated eigenvalues $\lambda_1 = \lambda_2 < 0$	Stable
Complex eigenvalues $\lambda = a + b\,j \quad b \neq 0$	
$a > 0$	Unstable oscillator: amplitude grows with time
$a = 0$	Stable: undamped oscillator
$a < 0$	Stable: damped oscillator

The eigenvectors give the **manifolds** of the system and the manifolds maybe stable or unstable. For the eigenvector (x_J, y_J), the manifold is given by the line from the Origin (0, 0) through the point (x_J, y_J).

Examples are presented to illustrate the main concepts of [2D] dynamics systems.

Example 1 Unstable multiple critical points

System: $\dot{x} = x \qquad \dot{y} = 0$

Dynamics:

$$f = x \quad df / dx = 1 \quad df / dy = 0$$

$$g = 0 \quad dg / dx = 0 \quad dg / dy = 0$$

$$\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \lambda_1 = 1 \quad \lambda_2 = 0$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \det(\mathbf{A}) = 0$$

Initial conditions (x0, y0)

(1.00, 1.00) (-1.00, -1.00) (-2.00, -2.00) (-1.50, -1.50)

(0.00, 0.00) (2.00, 2.00)

A matrix: a11 = 1.0 a12 = 0.0 a11 = 0.0 a12 = 0.0

Determinant A = 0.00

Eigenvalues Jacobian J = 1.00 0.00

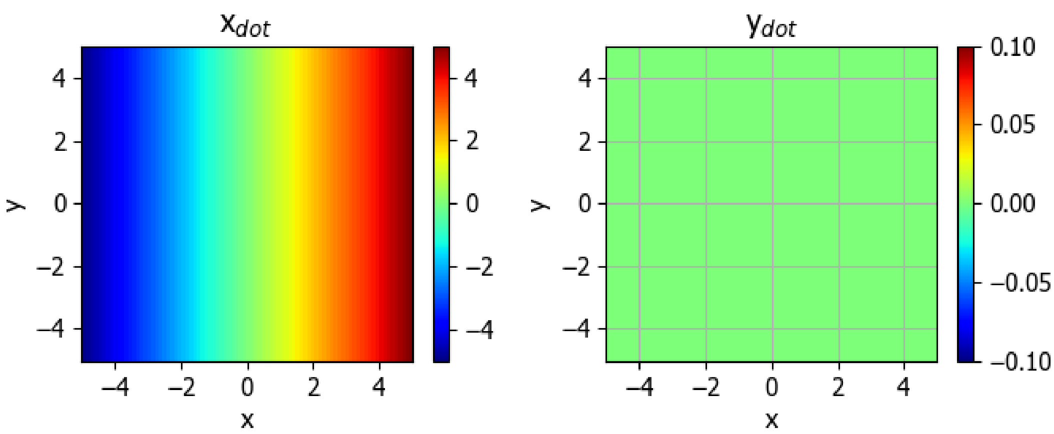

Fig. 1.1. [2D] view of the system equations.

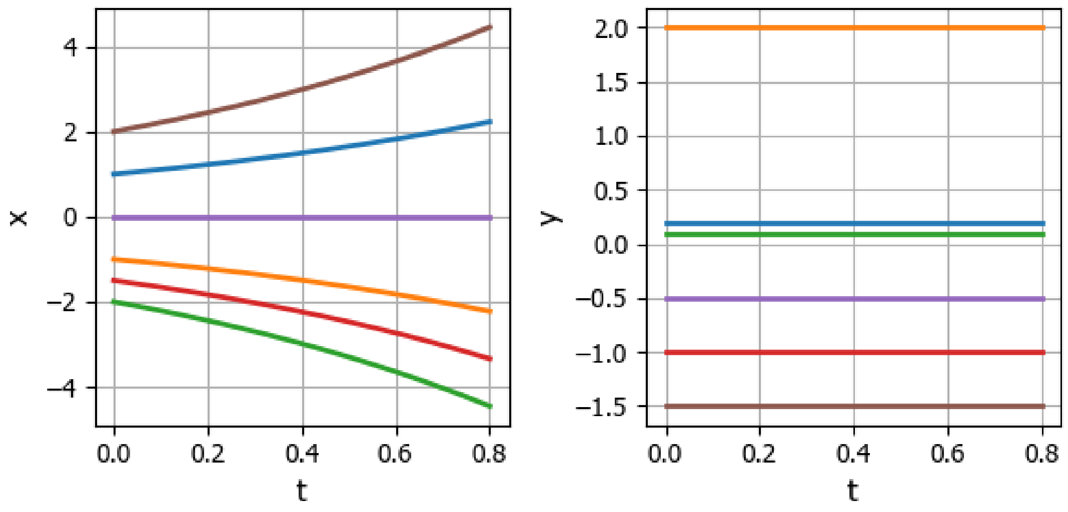

Fig. 1.2. Time evolution of the *x* and *y* parameters.

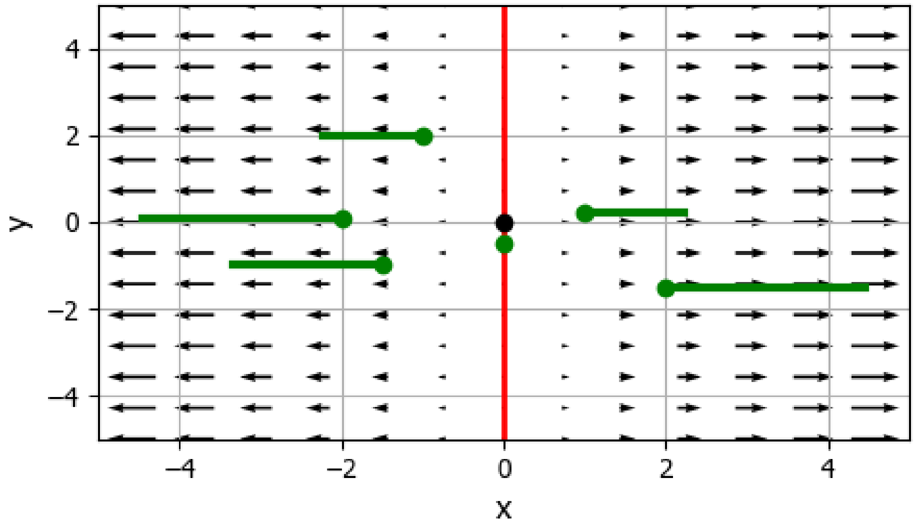

Fig. 1.3. Vector field: quiver plot. There are an infinite number of critical points lying along the ***y*-axis**. Six trajectories (orbits) in phase space for a time interval $\Delta t = 0.40$. The **black dot** is the single equilibrium point of the system at the Origin (0,0), and the **green dots** are for the different initial conditions ($x(0)$, $y(0)$). The length of the trajectory is shorter when the initial conditions are nearer the equilibrium point at the Origin.

Solutions of the ODEs are $x(t)$ and $y(t)$.

$y(0) \neq 0 \quad t \to \infty \quad x(t) \to \pm\infty \quad y(t) = y(0)$ where $x(t)$ changes exponentially.

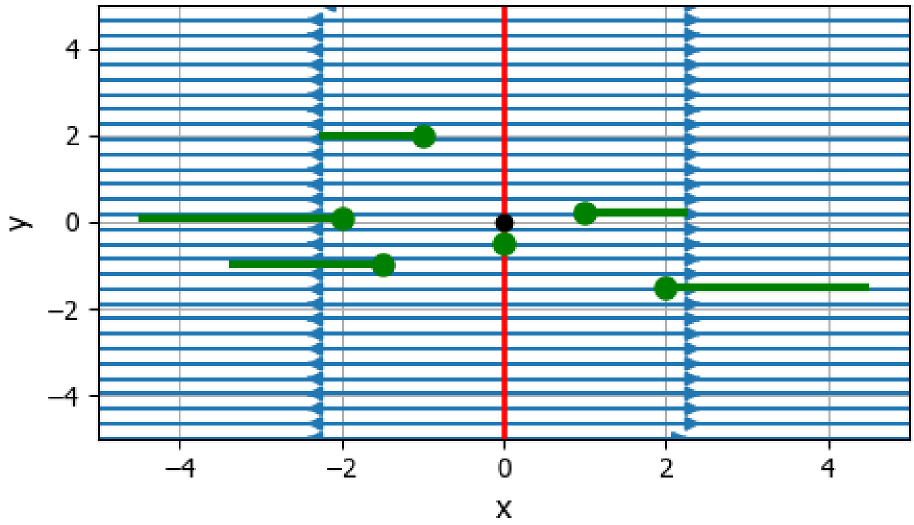

Fig. 1.5. Vector field: streamplot. There are an infinite number of critical points lying along the ***y*-axis**.

The determinant is zero $\det(\mathbf{A}) = 0$ and the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0$. One eigenvalue is real and positive and the other is zero. Therefore, the equilibrium point at the Origin (0,0) is an **unstable node**.

The critical points are found by solving the equations

$$\dot{x} = 0 \qquad \dot{y} = 0$$

which has the solution $x = 0$, y = constant. Thus, there are an infinite number of critical points lying along the y-axis. The direction field has gradient given by

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = 0 \qquad x \neq 0$$

This implies that the direction field is horizontal for points not on the y-axis. The direction vectors may be determined from the equation $\dot{x} = x$ since if $x > 0$, then $\dot{x} > 0$, and the trajectories move in $+x$ direction and if $x < 0$, then $\dot{x} < 0$, and trajectories move in the $-x$ direction.

Example 2 Points along the y-axis are stable nodes

System: $\dot{x} = -x \qquad \dot{y} = 0$

Dynamics:

$$f = x \quad df / dx = -1 \quad df / dy = 0$$

$$g = 0 \quad dg / dx = 0 \quad dg / dy = 0$$

$$\mathbf{J} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \quad \lambda_1 = -1 \quad \lambda_2 = 0$$

$$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \quad \det(\mathbf{A}) = 0$$

Initial conditions (x0, y0)
(1.00, 1.00) (-1.00, -1.00) (-2.00, -2.00) (-1.50, -1.50)
(0.00, 0.00) (2.00, 2.00)
A matrix: a11 = -1.0 a12 = 0.0 a11 = 0.0 a12 = 0.0
Determinant A = 0.00
Eigenvalues Jacobian J = -1.00 0.00

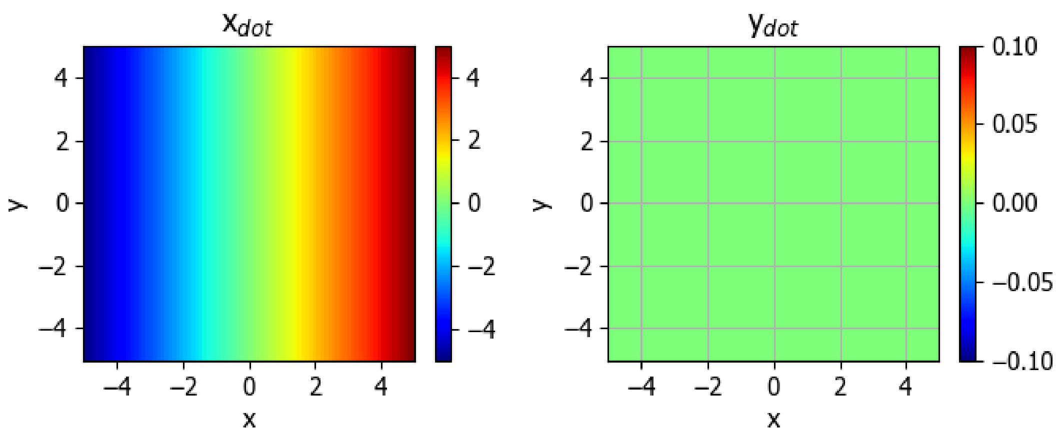

Fig. 2.1. [2D] view of the system equations.

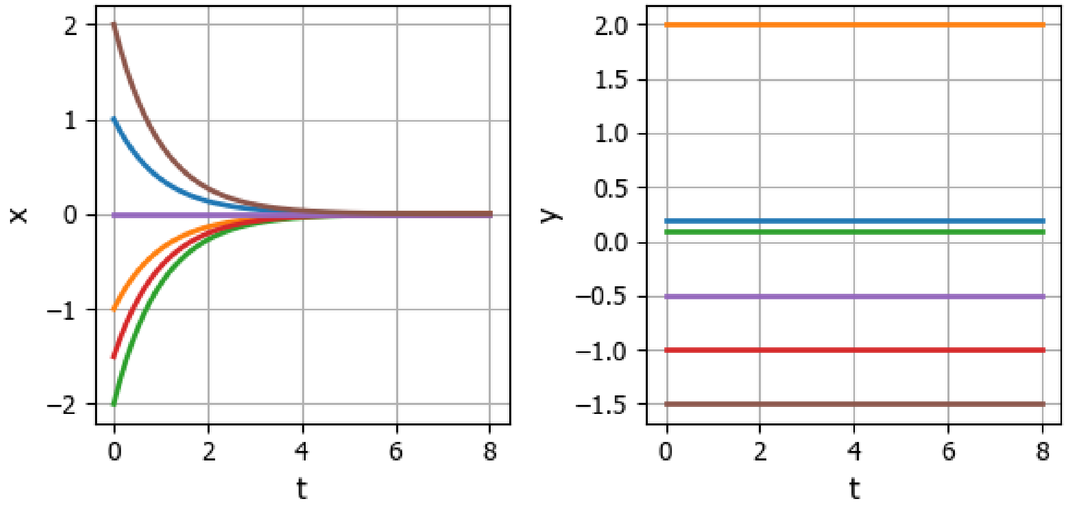

Fig. 2.2. Time evolution of the *x* and *y* parameters.

Six trajectories (orbits) in phase space for a time interval $\Delta t = 0.40$. The **black dot** is the equilibrium point of the system at the Origin (0,0), and the **green dots** are for the different initial conditions ($x(0)$, $y(0)$). Solutions of the ODEs are $x(t)$ and $y(t)$.

For all initial conditions: $t \to \infty \quad x(t) \to 0 \quad y(t) = y(0)$.

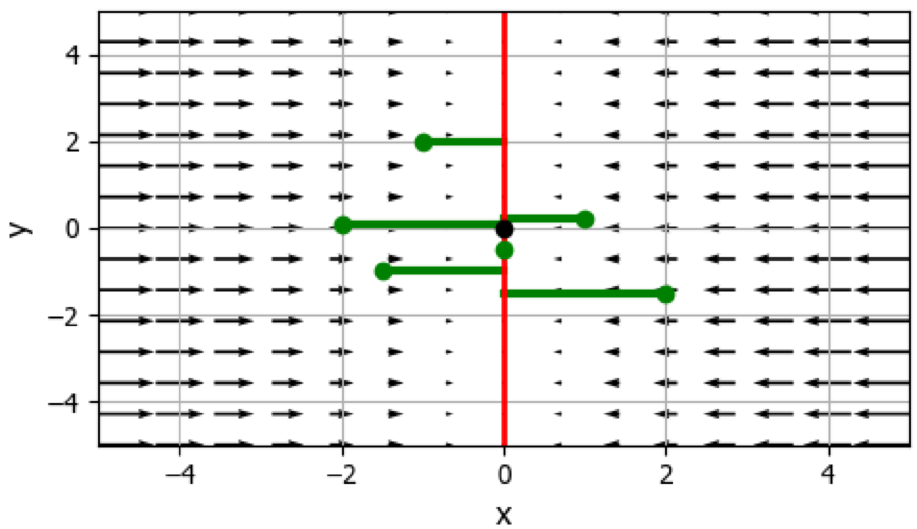

Fig. 2.4. Vector field: quiver plot. There are an infinite number of critical points lying along the ***y*-axis**.

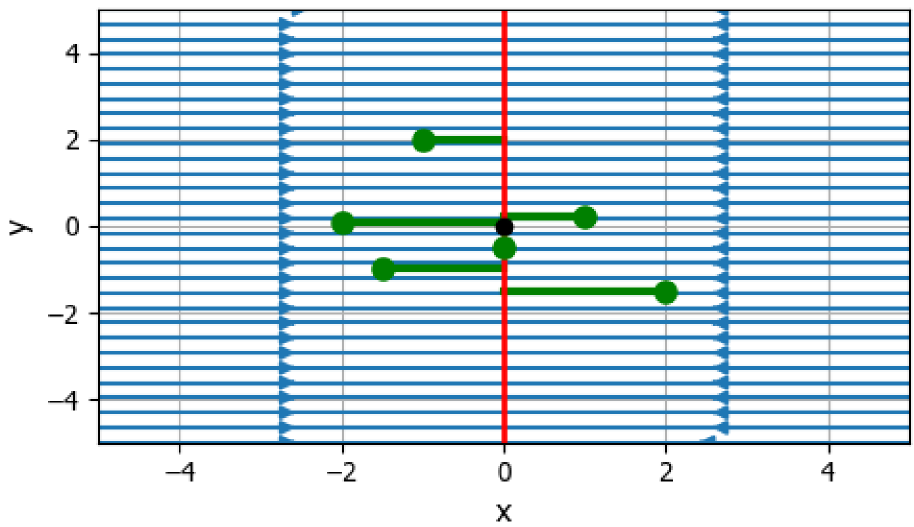

Fig. 2.5. Vector field: streamplot. There are an infinite number of critical points lying along the ***y*-axis**. Six trajectories (orbits) in phase space for a time interval $\Delta t = 0.40$. The **black dot** is the equilibrium point of the system at the Origin (0,0), and the **green dots** are for the different initial conditions ($x(0)$, $y(0)$). Solutions of the ODEs are $x(t)$ and $y(t)$. For all initial conditions: $t \to \infty \quad x(t) \to 0 \quad y(t) = y(0)$.

The determinant is zero $\det(\mathbf{A}) = 0$ and the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 0$. One eigenvalue is real and negative and the other is zero. Therefore, there is an infinite number of equilibrium points along the y-axis ($x = 0$). All trajectories, head towards $x = 0$ with constant y.

Example 3 Eigenfunctions and manifolds UNSTABLE

System: $\dot{x} = 2x + y \qquad \dot{y} = x + 2y$

A matrix: a11 = 2.0 a12 = 1.0 a11 = 1.0 a12 = 2.0

Determinant A = 3.00

Initial conditions (x0, y0)

(-1.00, 2.00) (-2.90, 1.31) (-1.50, -0.80) (1.00, -2.00)

(3.00, -1.50) (1.50, -0.50) (0.10, 0.10)

Eigenvalues Jacobian J1 = 3.00 J2 = 1.00

Eigenfunction Jacobian e0 = 1.00 1.00

Eigenfunctions Jacobian e1 = -1.00 1.00

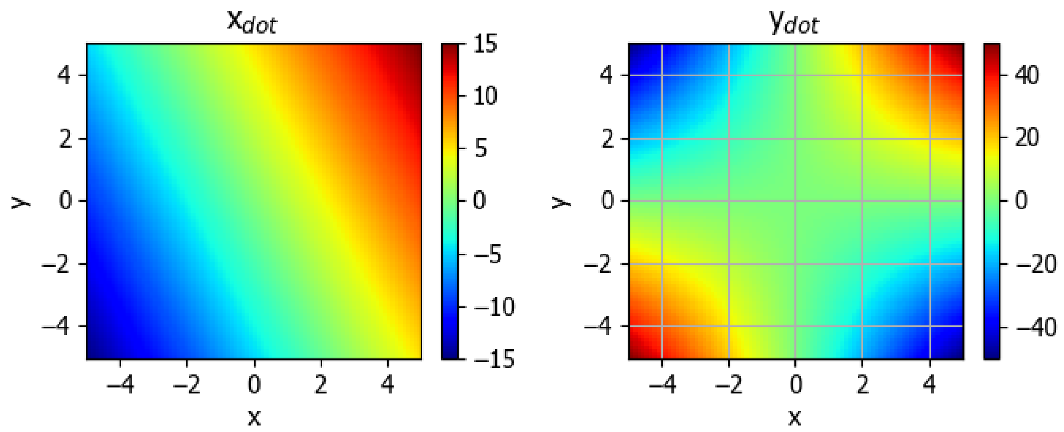

Fig. 3.1. [2D] view of the system equations.

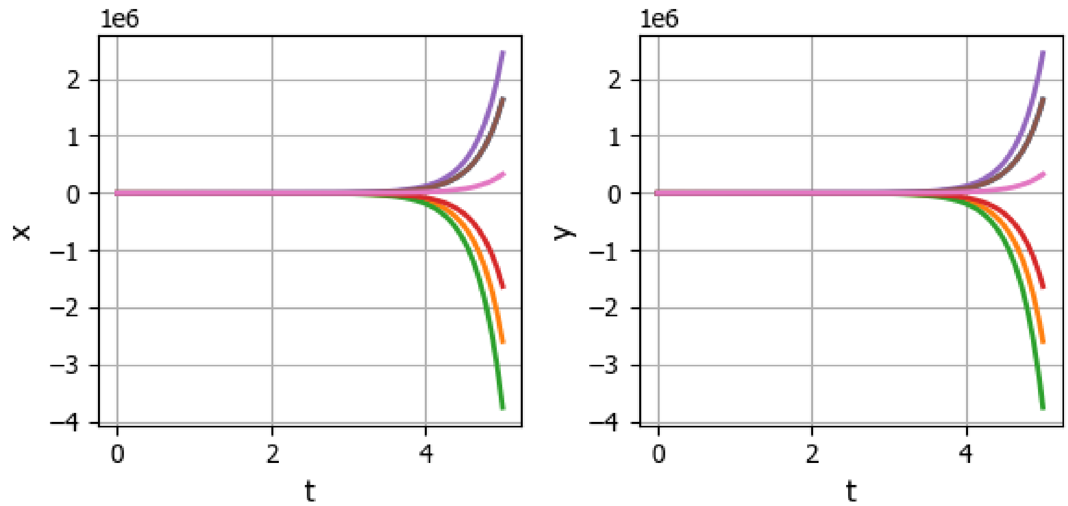

Fig. 3.2. Time evolution of the x and y parameters. All trajectories diverge to either $+\infty$ or $-\infty$.

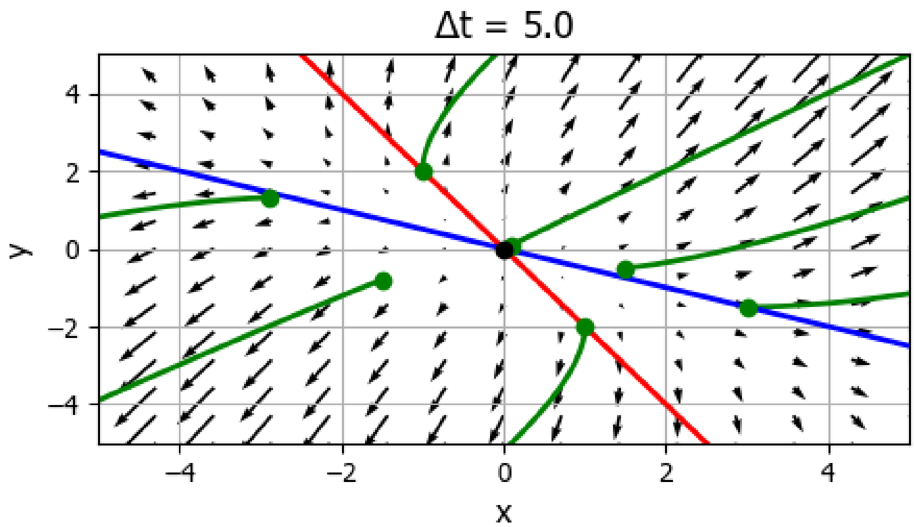

Fig. 3.3. Vector field: quiver plot. The red line is the x-nullcline $(\dot{x} = 0)$, and the blue line is the y-nullcline $(\dot{y} = 0)$.

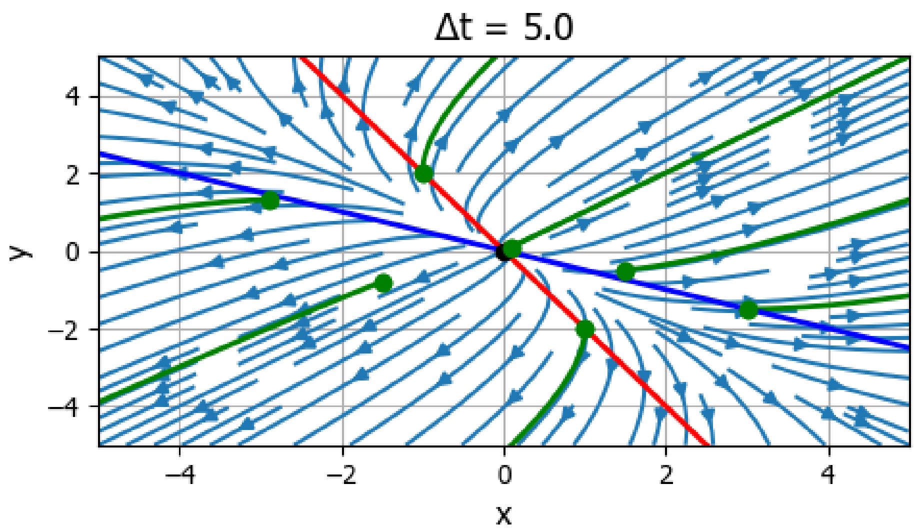

Fig. 3.4. Vector field: streamplot. The red line is the x-nullcline $(\dot{x} = 0)$, and the blue line is the y-nullcline $(\dot{y} = 0)$.

The intersection of the *x*-nullcline and the *y*-nullcline gives the **equilibrium point** (0,0). The determinant is non-zero $\det(\mathbf{A}) = 3 \neq 0$ and the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$. Both eigenvalues are real and positive. Therefore, the equilibrium point at the Origin (0,0) is an **unstable node**.

Eigenfunctions and manifolds

In 2x2 systems, eigenvectrors play a crucial role in understanding and simplifying the system's behaviour. They are used to define **invariant manifolds**, which are surfaces in the state space where the system's trajectories remain confined and trajectories starting on the manifold remain on it for all time.

The **eigenvalues** and **eigenvectors** (**eigenfunctions**) reveal the system's stability and the direction of its trajectories and gives information about the local behaviour around fixed points as shown in figure 3.5

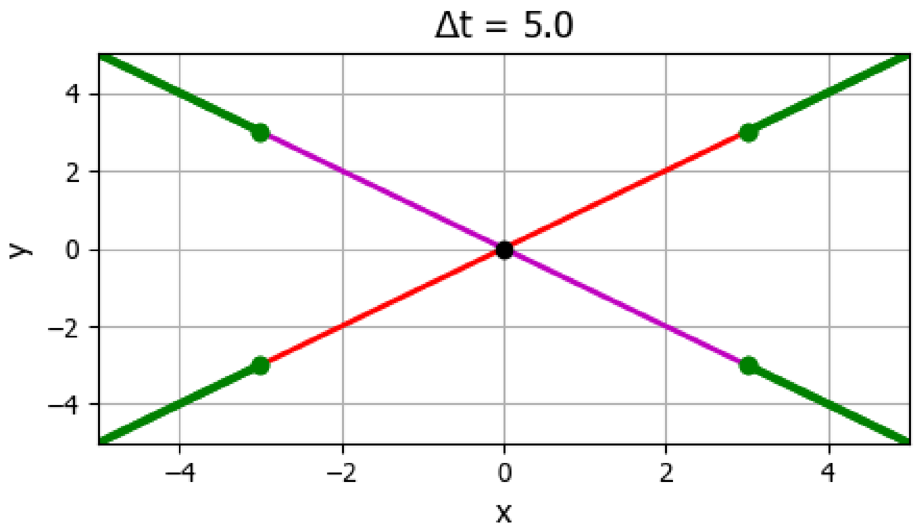

Fig. 3.5. The **manifolds** are defined by the eigenvectors **e0 (1,1)** and **e1 (-1,1)**. Both manifolds are **unstable**. One passes through $(0,\ 0)$ and $(1,\ 1)$ and the other through $(0,\ 0)$ and $(1, -1)$. All trajectories lying on the unstable manifolds diverge to either $+\infty$ or $-\infty$.

Example 4 SADDLE POINT

$$\dot{x} = -0.3x + 0.4y \qquad \dot{y} = -0.2x + 0.3y$$

A matrix: a11 = -0.3 a12 = 0.4 a11 = -0.2 a12 = 0.3

Determinant A = -0.01

Initial conditions (x0, y0)

(-1.00, 2.00) (-2.90, 1.31) (-1.50, -0.80) (1.00, -2.00)

(3.00, -1.50) (1.50, -0.50) (0.10, 0.10)

Eigenvalues Jacobian J1 = -0.10 J2 = 0.10

Eigenfunction Jacobian e0 = 2.00 1.00

Eigenfunctions Jacobian e1 = 1.00 1.00

Since one eigenvalue is **real** and **positive** and the other is **real** and **negative**, the critical point at the origin is a **saddle point**.

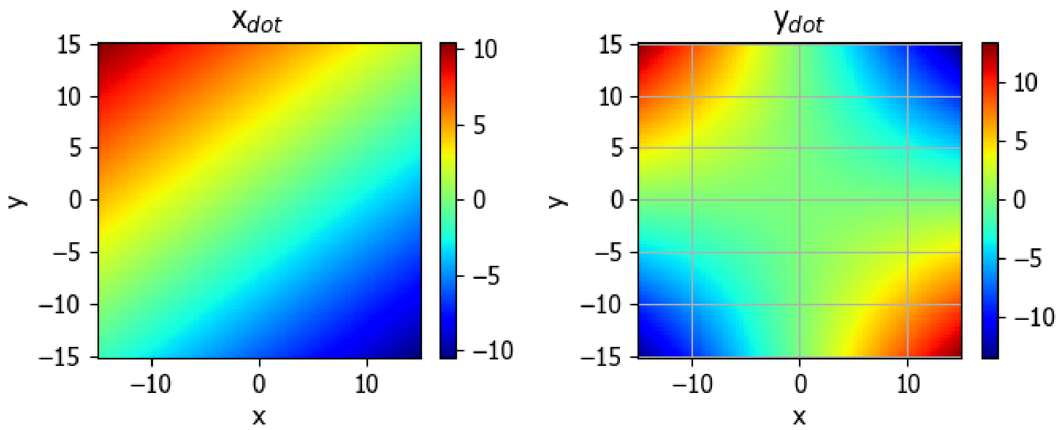

Fig. 4.1. [2D] view of the system equations.

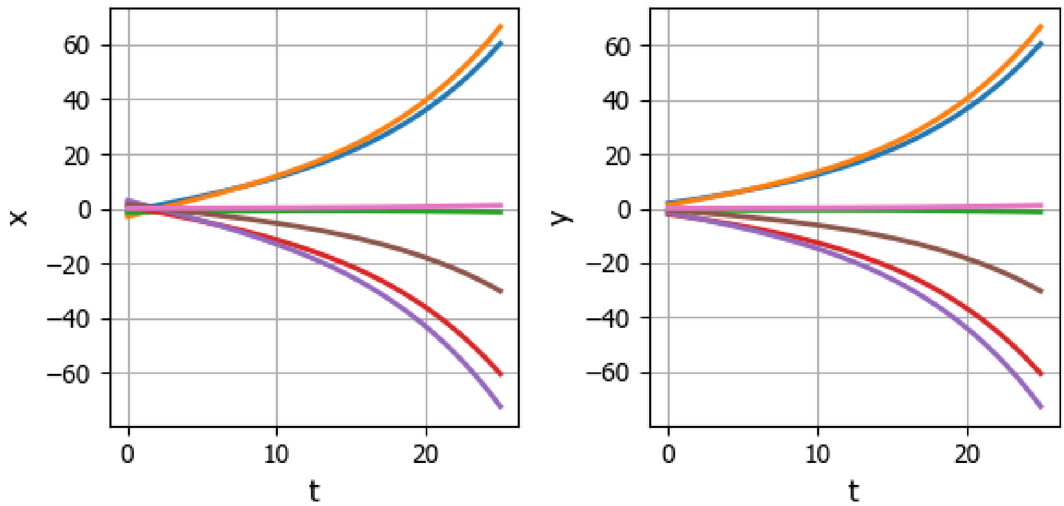

Fig. 4.2. Time evolution of the *x* and *y* parameters.

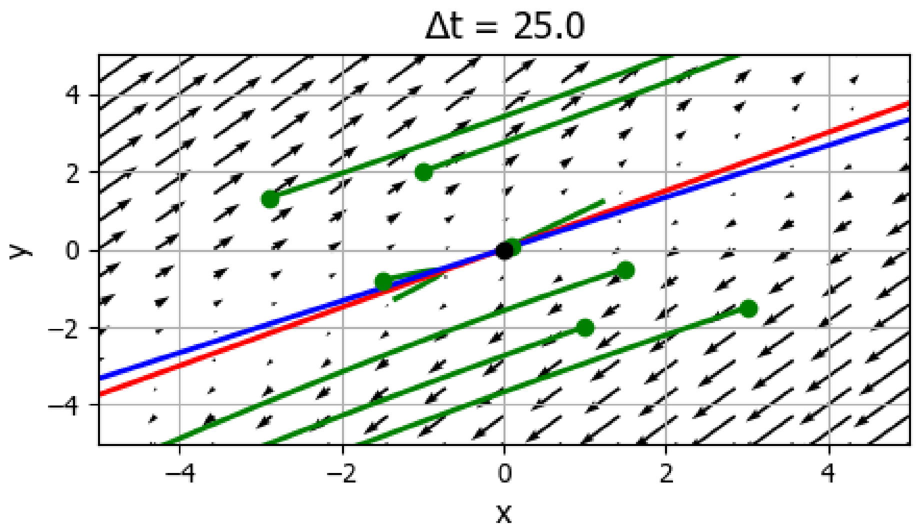

Fig. 4.3. Vector field: quiver plot. The red line is the x-nullcline $(\dot{x}=0)$, and the blue line is the y-nullcline $(\dot{y}=0)$.

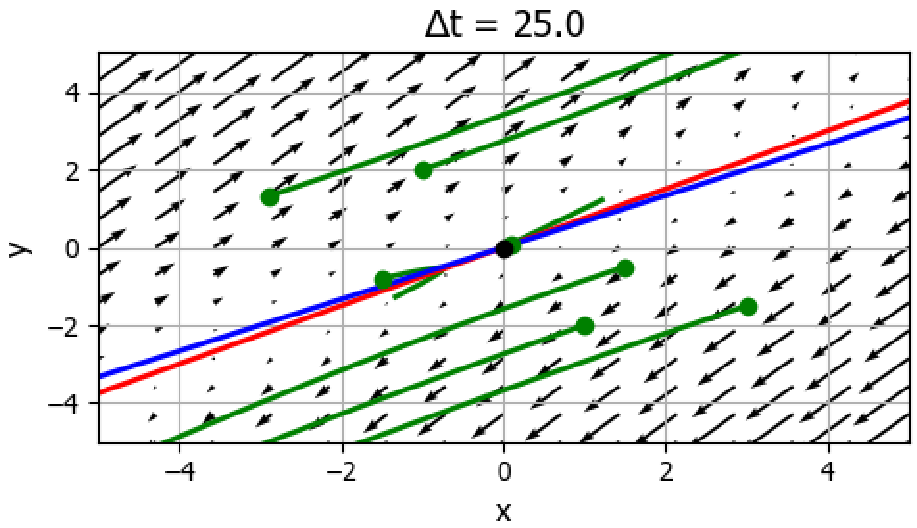

Fig. 4.4. Vector field: streamplot. The red line is the x-nullcline $(\dot{x}=0)$, and the blue line is the y-nullcline $(\dot{y}=0)$.

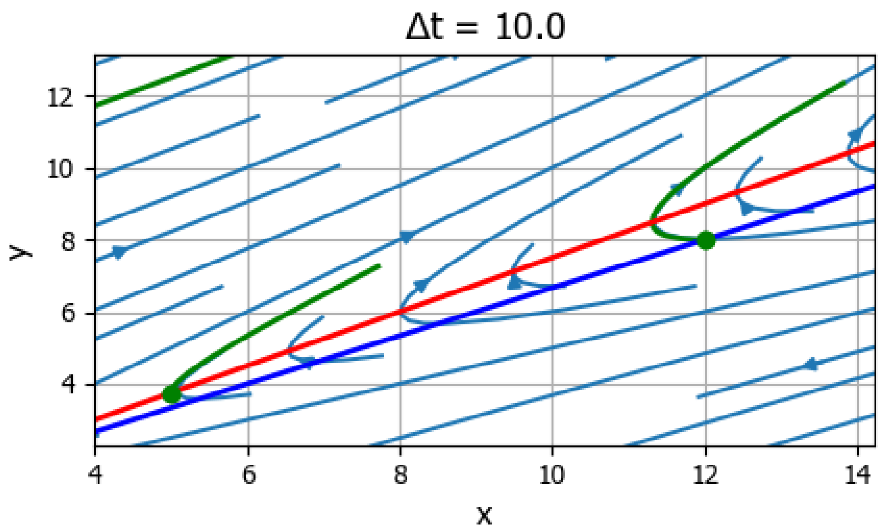

Fig. 4.5. Zoomed view of a streamplot. The initial condition point at (5.00,3.75) is located on the x-nullcline. Therefore, in the first-time step, the motion is only vertical. The other point is at (12.00, 8.00) and is on the y-nullcline. So, in the first-time step the motion must be in the horizontal direction.

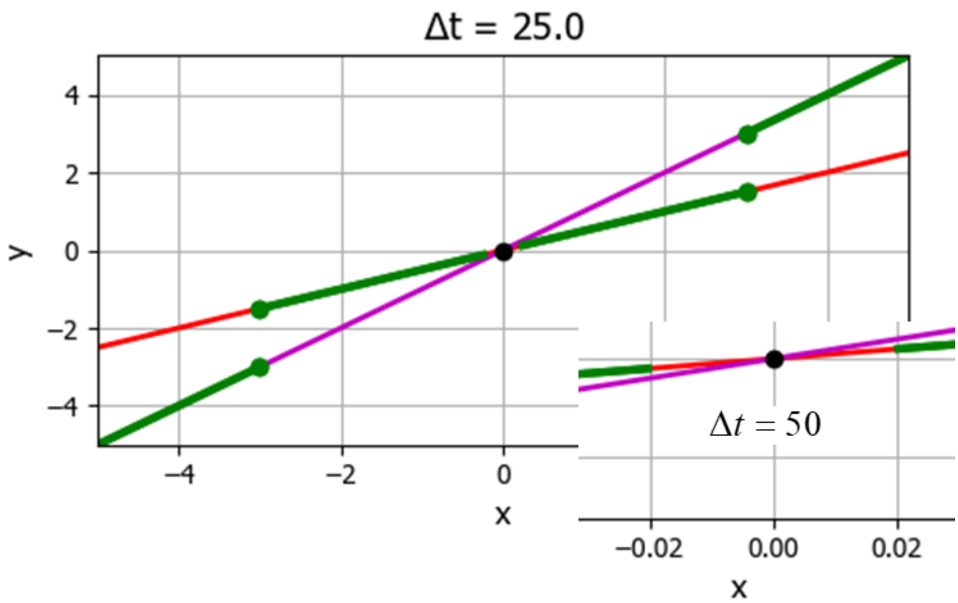

Fig. 4.6. The **stable** and **unstable** manifolds. The **manifolds** are defined by the eigenvectors **e0 (2,1)** and **e1 (1,1)**. The trajectories lying on the stable manifold tend to the origin as $t \to \infty$ but never reach it.

Example 5 STABLE FOCUS

$$\dot{x} = -1.0x - 1.0y \qquad \dot{y} = 1.0x - 1.0y$$

A matrix: a11 = -1.0 a12 = -1.0 a11 = 1.0 a12 = -1.0
Determinant A = 2.00
Initial conditions (x0, y0)
(1.00, 4.50) (-2.90, 1.31) (-3.00, -4.00) (1.00, -2.00)
(3.00, -1.50) (1.50, -0.50) (0.10, 0.10)
Eigenvalues (-1+1j) (-1-1j)
Eigenfunctions [-0.+1.j 0.-1.j] [1.-0.j 1.+0.j]

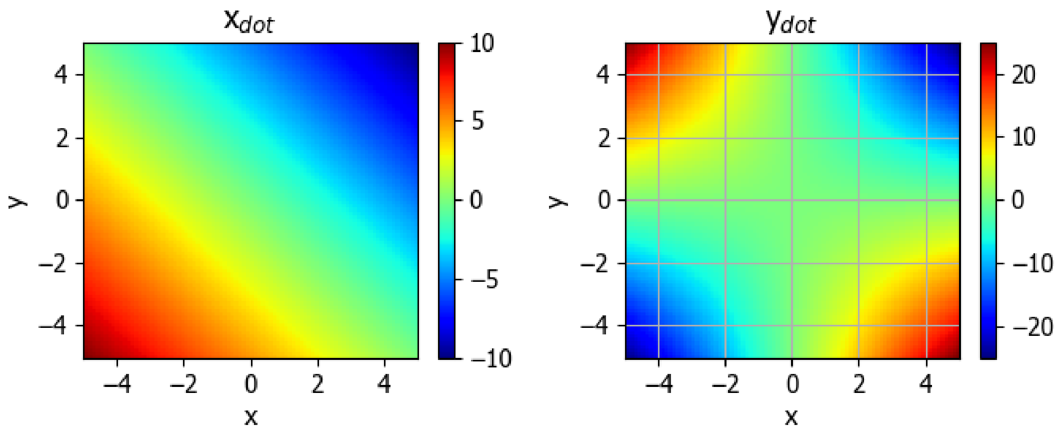

Fig. 5.1. [2D] view of the system equations.

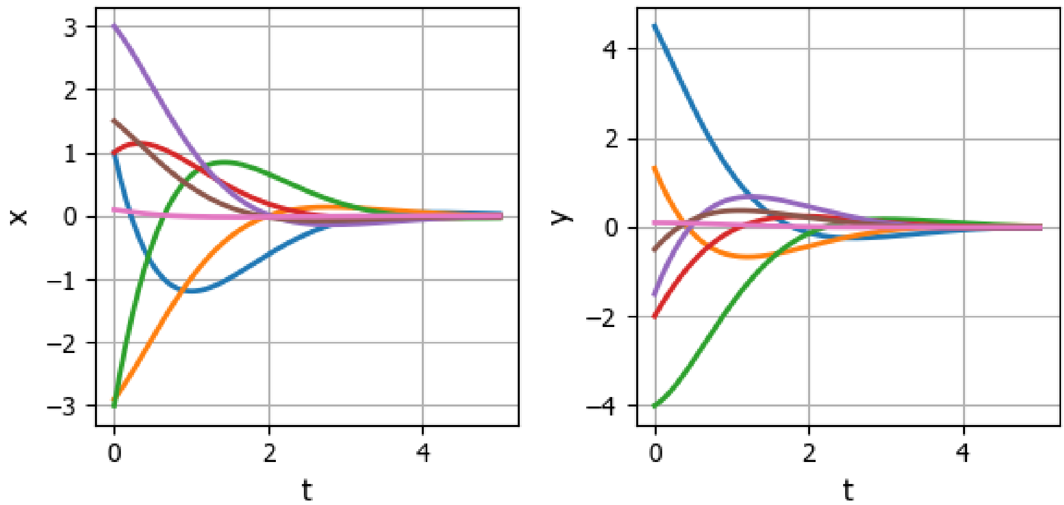

Fig. 5.2. Time evolution of the *x* and *y* parameters.

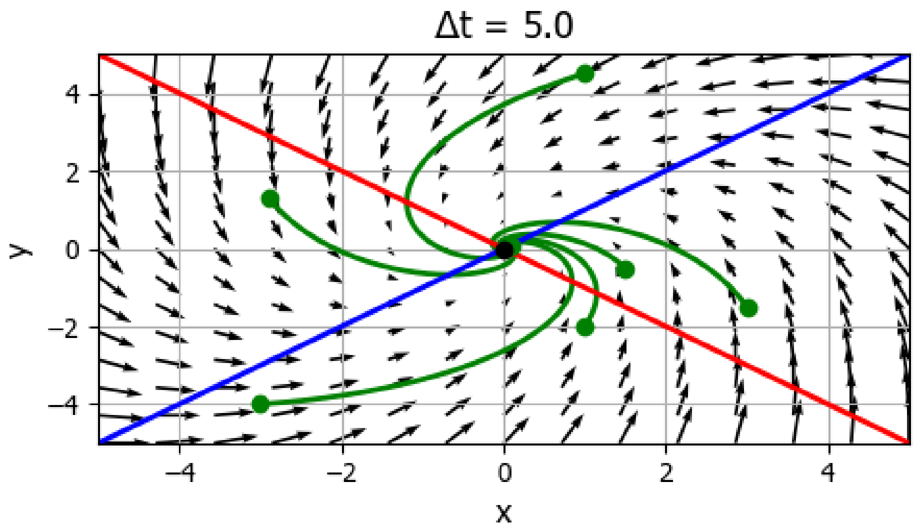

Fig. 5.3. Vector field: quiver plot. The red line is the x-nullcline $(\dot{x}=0)$, and the blue line is the y-nullcline $(\dot{y}=0)$.

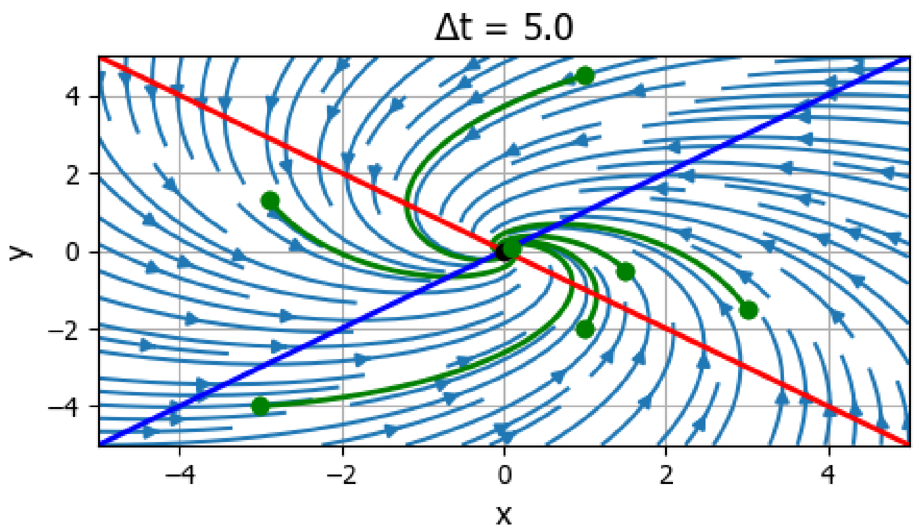

Fig. 5.4. Vector field: streamplot. The red line is the x-nullcline $(\dot{x}=0)$, and the blue line is the y-nullcline $(\dot{y}=0)$.

The eigenvalues and eigenvectors are complex solutions. The Origin is the only critical point and is a **stable focus**. The eigenvectors are complex and there are no real manifolds.

Example 6

$$\dot{x} = -2.0x \qquad \dot{y} = -4.0x - 2.0y$$

A matrix: a11 = -1.0 a12 = -1.0 a11 = 1.0   a12 = -1.0

Determinant A = 2.00

Initial conditions (x0, y0)

(1.00, 4.50) (-2.90, 1.31) (-3.00, -4.00) (1.00, -2.00)

(3.00, -1.50) (1.50, -0.50) (0.10, 0.10)

Eigenvalues (-1+1j) (-1-1j)

Eigenfunctions [-0.+1.j 0.-1.j] [1.-0.j 1.+0.j]

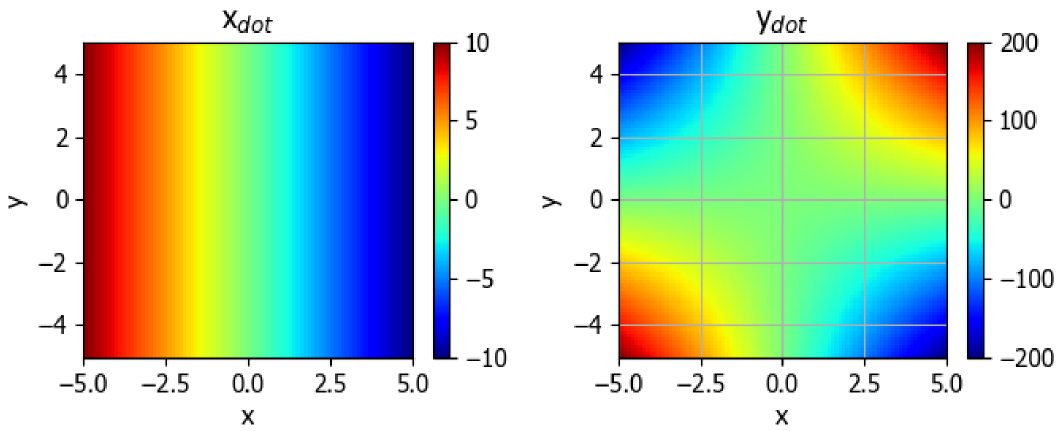

Fig. 6.1. [2D] view of the system equations.

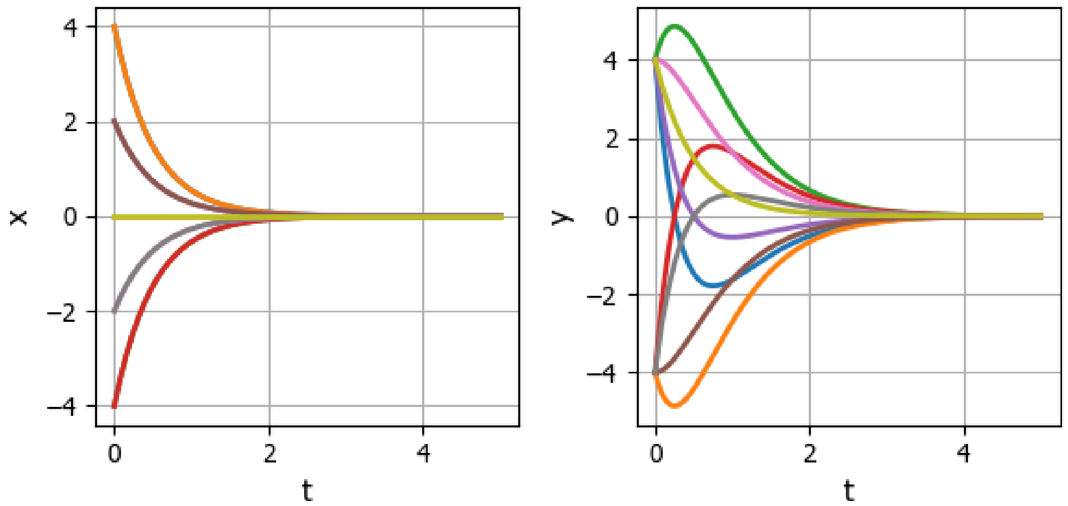

Fig. 6.2. Time evolution of the *x* and *y* parameters.

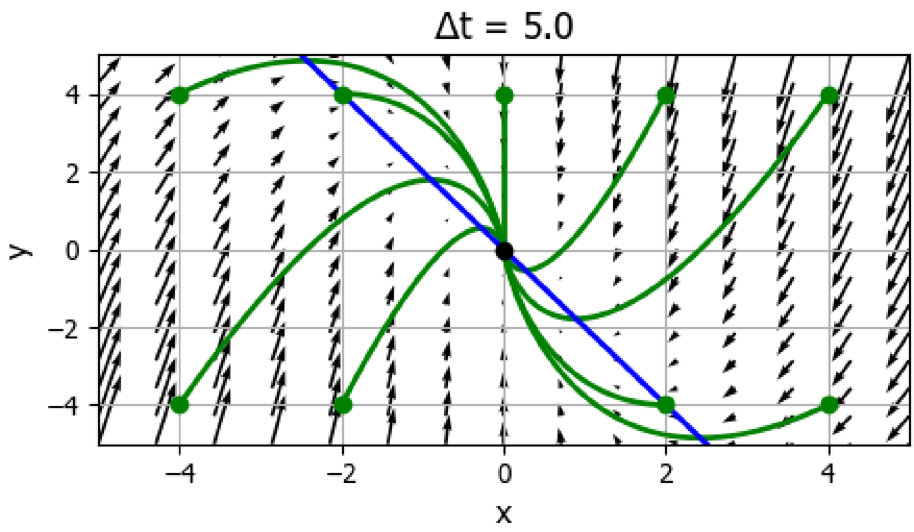

Fig. 6.3. Vector field: quiver plot. The red line is the x-nullcline $\left(\dot{x}=0\right)$, and the blue line is the y-nullcline $\left(\dot{y}=0\right)$.

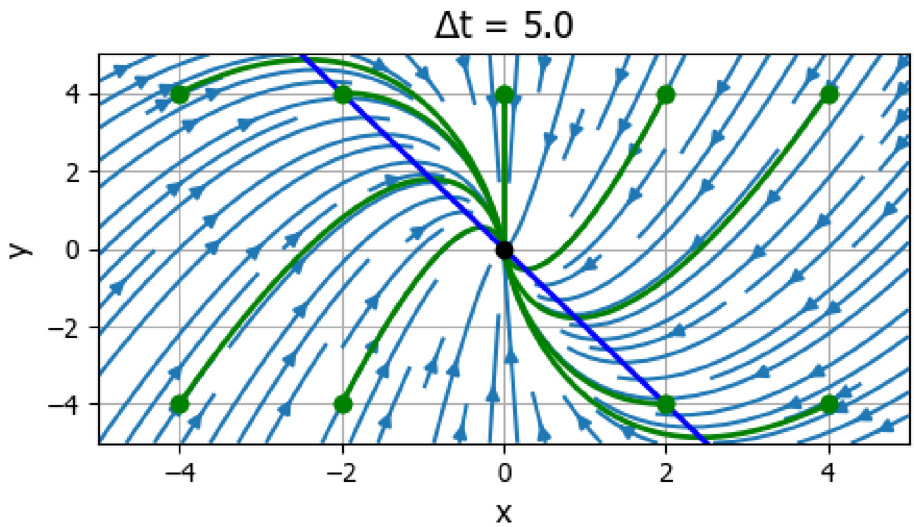

Fig. 6.4. Vector field: streamplot. The red line is the x-nullcline $\left(\dot{x}=0\right)$, and the blue line is the y-nullcline $\left(\dot{y}=0\right)$.

The Origin is the only critical point. In the phase portraits, the flow is horizontal on the line where $\dot{y}=0$ and hence on the line $y = -2x$. Trajectories which start on the y-axis remain there forever. There is only **one** linearly independent eigenvector. Therefore, the critical point is a **stable degenerate node**. The stable manifold is the y-axis.