

DOING PHYSICS WITH PYTHON

MOVING CHARGES IN ELECTRIC AND MAGNETIC FIELDS

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plasma002.py plasma004.py plasma006.py plasma008.py
plasma010.py

The Codes is used to calculate the trajectory of charged particles moving in a magnetic field and/or electric field. The equation of motion is solved using the Python function **solve_ivp**. The essential features of the method used are shown in the following Python Code:

```
import numpy as np
from numpy import pi, arange, linspace, zeros, array, ones,
sqrt, exp, tan, sin
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp
from scipy.signal import find_peaks
import time

tStart = time.time()
plt.close('all')
```

```

# %% Particle properties
q = 1.602e-19    # charge (C)
m = 1.67e-27     # mass (kg)

# Initial position and velocity
r0 = np.array([0.2, 0, 0.0])    # m
v0 = np.array([4e6, 4e6, 1e6])  # m/s

# Electric and magnetic fields
E = np.array([0.0, 0.0, 0.0])   # N/C
B = np.array([0.0, 0.0, 0.4])   # T

# %% Equations of motion
def equations(t, S):
    r = S[:3]    # position
    v = S[3:]    # velocity
    Bs = B #np.array([0,0,2])
    #if r[2] > 30: Bs = np.array([0,0,0])
    # Lorentz force
    F = q * (E + np.cross(v, Bs))
    # Acceleration
    a = F / m
    return np.concatenate((v, a))

# Initial state: [x, y, z, vx, vy, vz]
S0 = np.concatenate((r0, v0))

# Time interval
t1 = 0
t2 = 1e-6
N = 10000

t_eval = np.linspace(t1, t2, N)

```

```

# Solve ODE
solution = solve_ivp(
    equations,
    (t1, t2),
    S0,
    t_eval=t_eval,
    rtol=1e-8,
    atol=1e-10)

# Extract position, velocities and kinetic energy
x = solution.y[0]
y = solution.y[1]
z = solution.y[2]

vx = solution.y[3]
vy = solution.y[4]
vz = solution.y[5]

v = (vx**2+vy**2+vz**2)**0.5
K = 0.5*m*v**2

### Find period / frequency / radius
peaks, _ = find_peaks(y)
peak_times = t_eval[peaks]
periods = np.diff(peak_times)
Tavg = np.mean(periods)    # [s]
favg = 1/Tavg              # [z]
R = (max(x)-min(x))/2
# Theoretical values
Bz = B[2]
RC = m*max(v)/(q*Bz)
wC = q*Bz/m
fC = wC/(2*pi)
TC = 1/fC

```

From the solution of the equation of motion the position and velocity as a function of time are known. For the motion in a constant magnetic field in the Z-direction, the radius of orbit in an XY-plane and the cyclotron frequency can be estimated.

INTRODUCTION

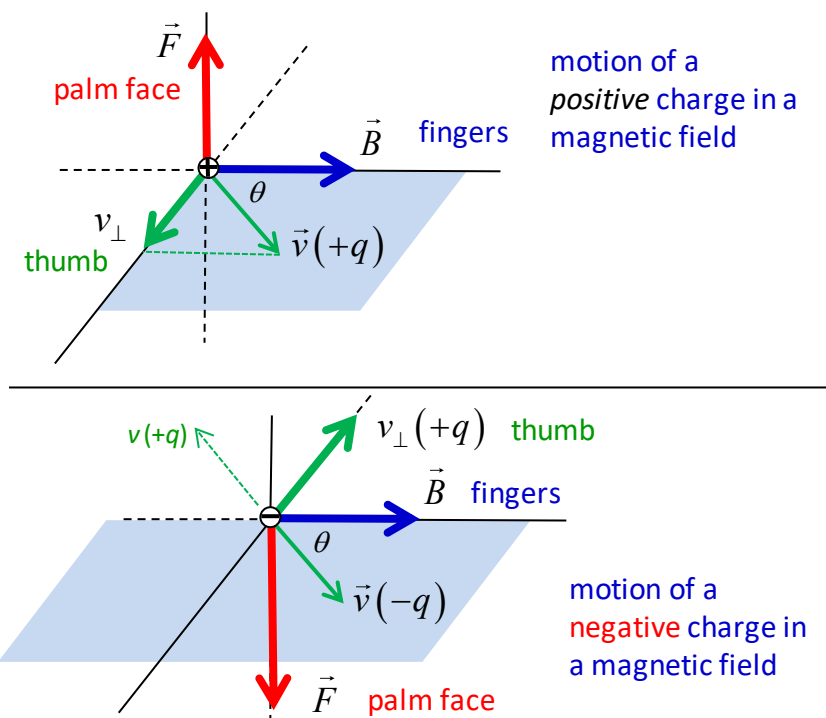
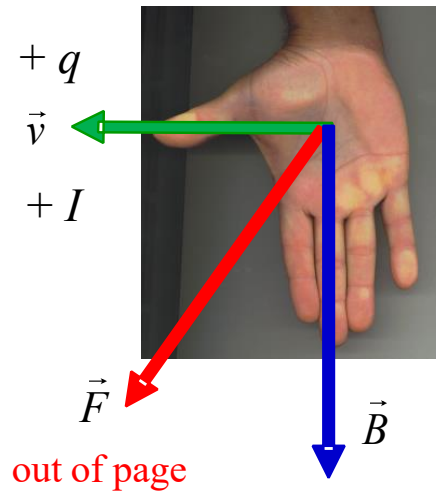
A charged particle of mass m and charge q will experience a force acting upon it in an electric field $\vec{E}$. Also, the charged particle will experience a magnetic force acting upon it when moving with a velocity $\vec{v}$ in a magnetic field $\vec{B}$.

If the charged particle is moving in the presence of both an electric field and magnetic field, the force $\vec{F}$ acting on it is called the **Lorentz force**

$$(1) \quad \vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \quad \text{Lorentz force}$$

If the charged particle is stationary ($\vec{v} = 0$), the force depends only of the electric field. The direction of the electric force is in the same direction as the electric field if $q > 0$ and the electric force is in the opposite direction to the electric field if $q < 0$.

When a charged particle is moving only in a magnetic field, the direction of the magnetic force is at right angles to both the direction of motion and the direction of the magnetic field as given by the right-hand palm rule.



The magnetic force $q\vec{v}\times\vec{B}$ is always perpendicular to the velocity $\vec{v}$ and so it does no work on the particle and does not change its speed or kinetic energy. The magnetic force only changes the direction of motion, which tends to make the charged particle go in a circle or in a helix.

The charged particle will move in a circular path of radius R in a uniform magnetic field when $\vec{v}$ and $\vec{B}$ are perpendicular to each other. In this situation, the centripetal force is simply the magnetic force

$$\frac{mv^2}{R} = |qvB|$$
$$(2) \quad R = \left| \frac{mv}{qB} \right| \quad \text{Larmor radius}$$

The Larmor radius of the circular orbit depends on the momentum of the particle, its charge and the strength of the magnetic field.

A charged particle trapped in a circular orbit by the **B** field displays a characteristic **cyclotron frequency** ω_c (or f_c). The cyclotron period T_c is the time for one revolution

(3a)

$$T_C = \left| \frac{2\pi R}{v} \right| = \left| \frac{2\pi m}{qB} \right| \quad f_C = \frac{1}{T_C} = \left| \frac{qB}{2\pi m} \right| \quad \omega_C = 2\pi f_C$$

$$(3b) \quad \omega_C = \left| \frac{q}{m} B \right| \quad \text{cyclotron frequency}$$

The period T_C and the cyclotron frequency ω_C are both independent of the velocity of the particle. This fact is made use of in many applications.

If the charge particle's velocity is neither parallel nor perpendicular to the magnetic field, the trajectory of the particle is a **helix**. If the **B** field is parallel to the Z axis and the initial velocity is parallel to the XZ plane, then, the particle moves in the Z direction with uniform speed v_z while it continues to go in circles of radius $R = m v_{xy} / q B$ where the v_{xy} is the velocity of the charged particle in an XY plane.

If there is region of crossed **E** and **B** fields ($\vec{E} \perp \vec{B}$) then the magnitudes of the fields can be adjusted so that a particle can move without any deflection when

$$(4) \quad v = \frac{E}{B} \quad \text{velocity } v \text{ independent of the mass } m$$

In the region of the crossed **E** and **B** fields, the trajectory will be a **cycloid** if the speed is not too great. The charged particle starts from rest, then, it tends to migrate in the direction of the vector $\vec{E} \times \vec{B}$.

The motion of charged particle, usually electrons or positive ions, under the action of **E** and **B** fields is the basis of many fundamental experiments in physics, for example, magnetic focusing, measurement of charge to mass ratio (q / m) in a mass spectrometer, cyclotron and magnetron.

EQUATION OF MOTION

The equation of motion of the charged particle moving in the **E** and **B** fields can be found from the Lorentz force and Newton's Second law of Motion and from which we can give the acceleration $\vec{a}$ of the particle at any instant as

$$(5) \quad \vec{a} = \frac{q}{m} (\vec{E} + \vec{v} \times \vec{B})$$

We will only consider trajectories of a charged particle in regions of uniform crossed **E** and **B** fields ($\vec{E} \perp \vec{B}$) where the **E** field is in the direction of the Y axis and the **B** field in the direction of the Z axis. The vectors for the **E** field and **B** field and their Cartesian components are

$$\vec{E}(0, E, 0) \quad \text{and} \quad \vec{B}(0, 0, B)$$

The description of the trajectory is given its terms of a charged particle displacement, velocity and acceleration vectors and their Cartesian components

$$\text{Acceleration} \quad \vec{a} (a_x, a_y, a_z)$$

$$\text{Velocity} \quad \vec{v} (v_x, v_y, v_z)$$

$$\text{Displacement} \quad \vec{s} (x, y, z)$$

The cross product $\vec{v} \times \vec{B}$ can be expressed as

$$\vec{v} \times \vec{B} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{pmatrix}$$

The equation of motion is easily solved in Python using the function **solve_ivp**.

SIMULATIONS

Simulation 1 Uniform magnetic field

$$B_x = 0, B_y = 0, B_z = \text{constant}$$

plasma002.py

Python (Spyder) Console Summary

INPUTS

$$m = 1.67\text{e-}27 \text{ kg}$$

$$q = 1.60\text{e-}19 \text{ kg}$$

$$x0 = 0.20 \text{ m} \quad y0 = 0.00 \text{ m} \quad z0 = 0.00 \text{ m}$$

$$vx0 = 4.0\text{e+}06 \text{ m/s} \quad vy0 = 4.0\text{e+}06 \text{ m/s} \quad vz0 = 1.0\text{e+}06 \text{ m/s}$$

$$Ex = 0.00 \text{ N/C} \quad Ey = 0.00 \text{ N/C} \quad Ez = 0.00 \text{ N/C}$$

$$Bx = 0.00 \text{ T} \quad By = 0.00 \text{ T} \quad Bz = 0.40 \text{ T}$$

$$\text{Simulation time } t2 = 1.00\text{e-}06 \text{ s}$$

$$\text{Time steps } N = 10000$$

OUTPUTS

$$\text{Radius: } R_{\text{simulation}} = 0.147 \text{ m} \quad R_{\text{theory}} = 0.147 \text{ m}$$

$$\text{Cyclotron period: } T_{\text{simulation}} = 1.64\text{e-}07 \text{ s}$$

$$T_{\text{theory}} = 1.64\text{e-}07 \text{ s}$$

$$\text{Cyclotron frequency: } f_{\text{simulation}} = 6.11\text{e+}06 \text{ Hz}$$

$$f_{\text{theory}} = 6.11\text{e+}06 \text{ Hz}$$

$$\text{KE is constant: } KE = 2.76\text{e-}14 \text{ J}$$

$$\text{Execution time} = 0.58 \text{ s}$$

The simulation results for both the radius and cyclotron frequency were identical to the theoretical predictions.

The results of the simulation are shown in the following figures.

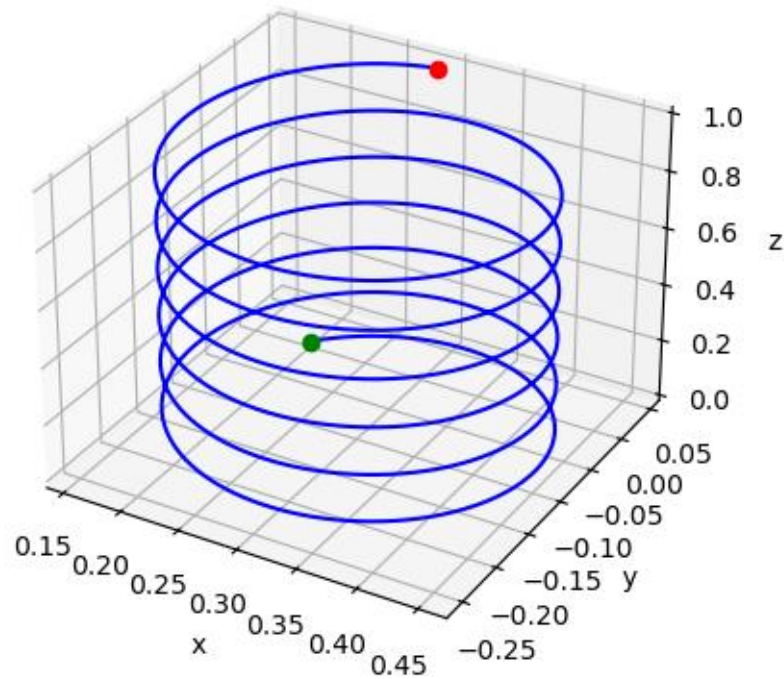


Fig. 1.1. The [3D] trajectory of the proton. The path of the proton is in the shape of a helix. The proton cycles around the magnetic fields lines which point in the +Z-direction.

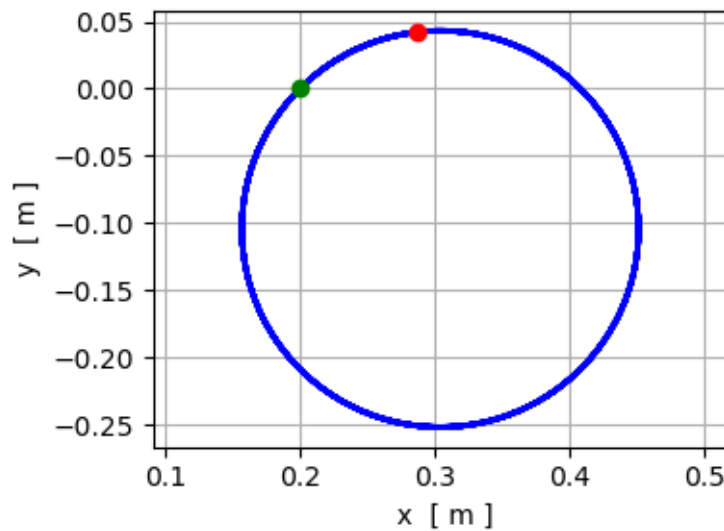


Fig. 1.2. The circular path of the proton in an XY plane. The radius of the circular path is $R = 0.147$ m.

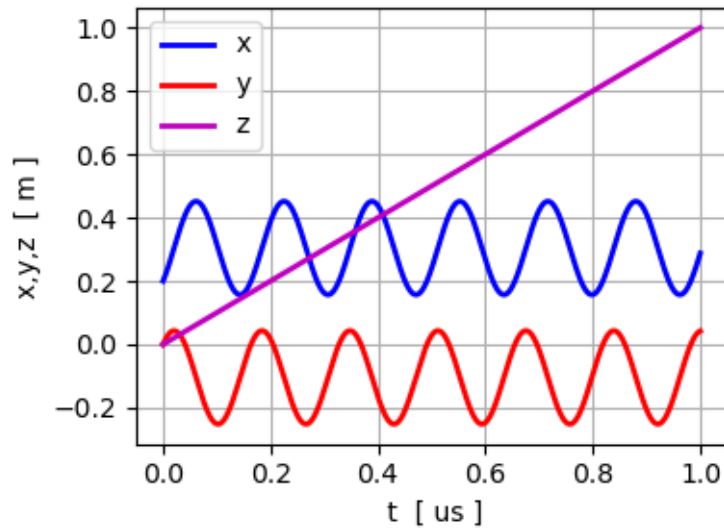


Fig. 1.3. Displacement vs time graph. The X and Y components of the displacement vary sinusoidally with time. The charged particle executes simple harmonic motion in the X and Y directions. The proton moves with a constant velocity in the Z-direction. The cyclotron period and frequency are: $T_C = 1.64 \times 10^{-7}$ s and $f_C = 6.11 \times 10^6$ Hz.

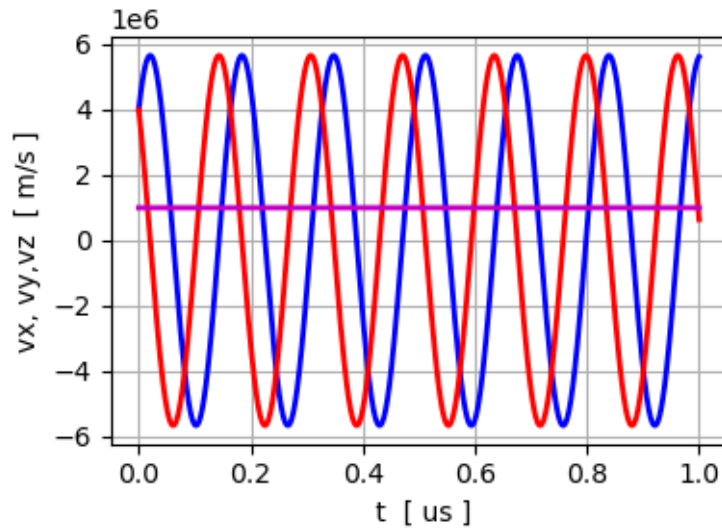


Fig. 1.4. Velocity vs time graph. The X and Y components of the velocity vary sinusoidally with time. The magnitude of the velocity v is constant. The magnetic force does zero work on the charged particle when it moves through the uniform **B** field.

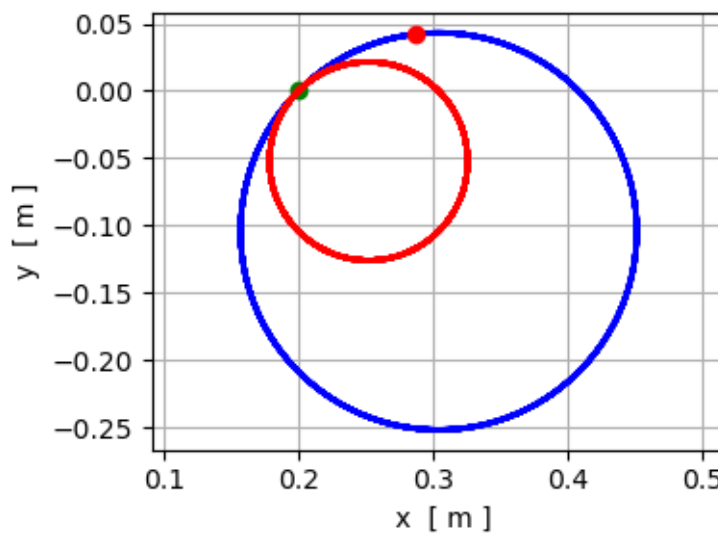


Fig. 1.5. Paths of the proton in an XY plane.

Blue curve $B = 0.40$ T Red curve $B = 0.80$ T

$$B_z = 0.40 \text{ T: } R = 0.147 \text{ m} \quad T_C = 16.4 \times 10^{-8} \text{ s} \quad f_C = 6.11 \times 10^6 \text{ Hz}$$

$$B_z = 0.80 \text{ T: } R = 0.074 \text{ m} \quad T_C = 8.19 \times 10^{-8} \text{ s} \quad f_C = 12.2 \times 10^6 \text{ Hz}$$

The larger the **B** field, the greater the strength of the magnetic force acting on the charged particle, hence, the smaller the radius of the circular orbit and higher orbital (cyclotron) frequency.

Motion of an electron `plasma003.py`

$$m_p / m_e = 1833 \text{ then } R_e = R_p / 1833 \quad T_e = T_p / 1833$$

INPUTS

$$m = 9.11 \times 10^{-31} \text{ kg}$$

$$q = 1.60 \times 10^{-19} \text{ kg}$$

$$x_0 = 2.00 \times 10^{-1} \text{ m} \quad y_0 = 0.00 \times 10^0 \text{ m} \quad z_0 = 0 \times 10^0 \text{ m}$$

$$v_{x0} = 4.0 \times 10^6 \text{ m/s} \quad v_{y0} = 4.0 \times 10^6 \text{ m/s} \quad v_{z0} = 1.0 \times 10^6 \text{ m/s}$$

$$E_x = 0.00 \text{ N/C} \quad E_y = 0.00 \text{ N/C} \quad E_z = 0.00 \text{ N/C}$$

$$B_x = 0.00 \text{ T} \quad B_y = 0.00 \text{ T} \quad B_z = 0.40 \text{ T}$$

$$\text{Simulation time } t_2 = 5.00 \times 10^{-10} \text{ s}$$

$$\text{Time steps } N = 10000$$

OUTPUTS

$$\text{Radius: } R_{\text{simulation}} = 8.04 \times 10^{-5} \text{ m} \quad R_{\text{theory}} = 8.04 \times 10^{-5} \text{ m}$$

$$\text{Cyclotron period: } T_{\text{simulation}} = 8.93 \times 10^{-11} \text{ s} \quad T_{\text{theory}} = 8.93 \times 10^{-11} \text{ s}$$

$$\text{Cyclotron frequency: } f_{\text{simulation}} = 1.12 \times 10^{10} \text{ Hz} \quad f_{\text{theory}} = 1.12 \times 10^{10} \text{ Hz}$$

$$\text{KE is constant: } KE = 1.50 \times 10^{-17} \text{ J}$$

$$\text{Execution time} = 0.35 \text{ s}$$

Simulation 2 Magnetic deflection `plasma004.py`

Magnetic fields are commonly used to control the path of charged particles. Figure 2.1 shows the trajectories of a proton launched with initial speed $u_x = 8.0 \times 10^6 \text{ m.s}^{-1}$ and $u_y = 0 \text{ m.s}^{-1}$ into uniform magnetic fields of various strengths. When $B_z > 0$ the $\mathbf{B}$ field is in the +Z direction (out of page) and $B_z < 0$ the $\mathbf{B}$ field is in the -Z direction (into page). The deflection of the particle is given by the right-hand rule. The greater the strength of the magnetic field, then the greater the deflection of the charged particle as it traverses the magnetic field.

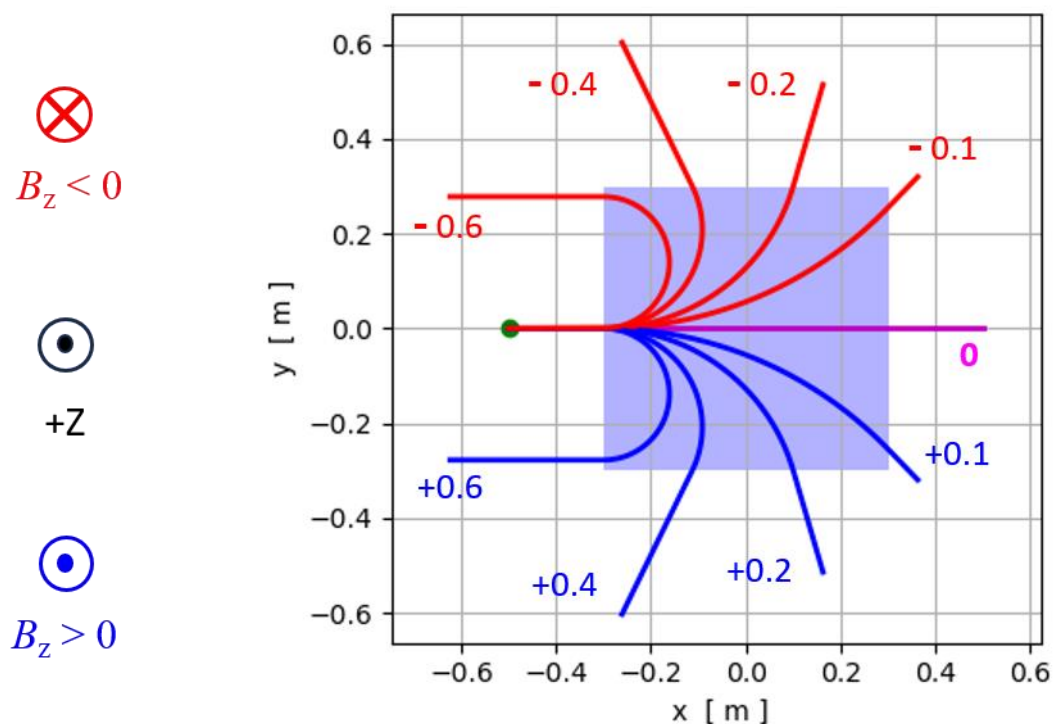


Fig. 2.1. The trajectories of a proton traversing a uniform magnetic field of different strengths. The numbers give the magnetic field strengths [T]. The simulation time was $t_2 = 0.12 \times 10^{-6} \text{ s}$.

The particle travels in a straight line when $B = 0$ and is deflected by the magnetic field when $B \neq 0$ which tends to cause the charged to move in a circular orbit. Figure 2.2 shows the displacement and velocity graphs.

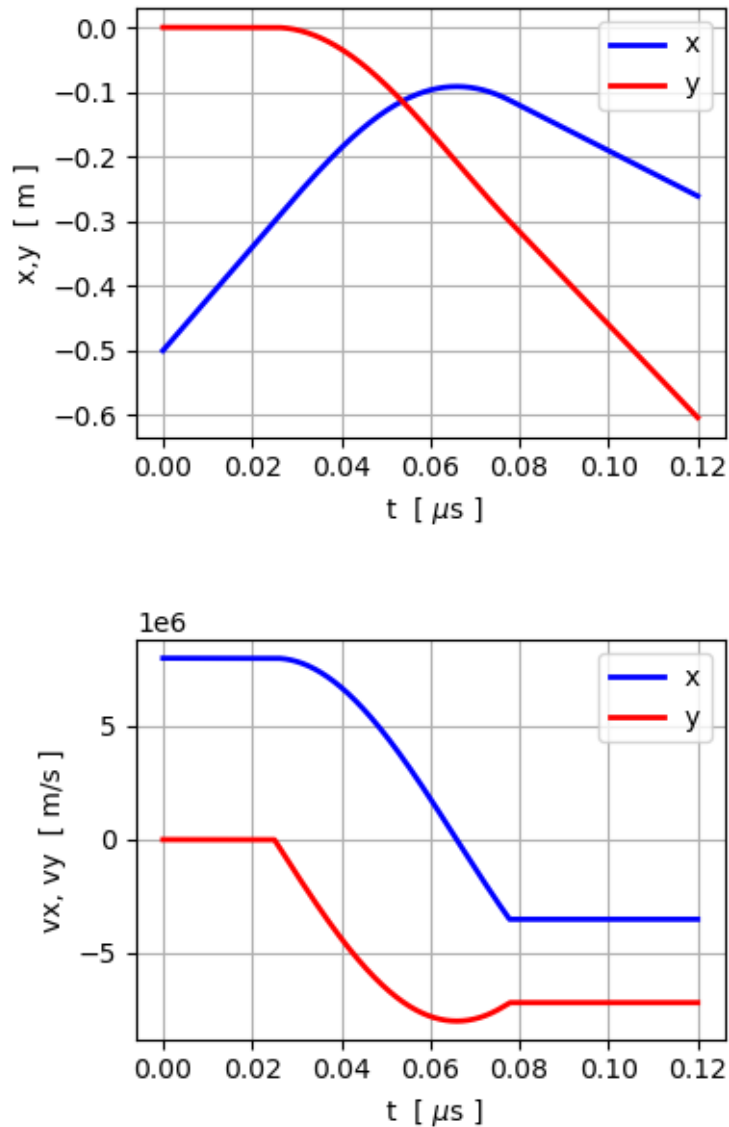


Fig. 2.2. Displacement and velocity time graph for the motion of the proton. When $B_z = 0$, the proton travels with a constant velocity. In the time interval from $0.025 \mu s$ to $0.078 \mu s$, $B_z > 0$ and the proton is accelerated as it changes direction.

The deflection of a charge particle in a constant magnetic field depends upon the mass the particle. This deflection is the basis of a mass spectrometer. The radius of curvature is

$$R = \frac{mv}{qB}$$

So, the larger the mass the greater the radius of curvature.

Figure 2.3 shows the deflection of a proton ($m = 1.67 \times 10^{-27}$ kg) and a deuteron ($m = 3.34 \times 10^{-27}$ kg) in the magnetic field $B_z = 0.10$ T.

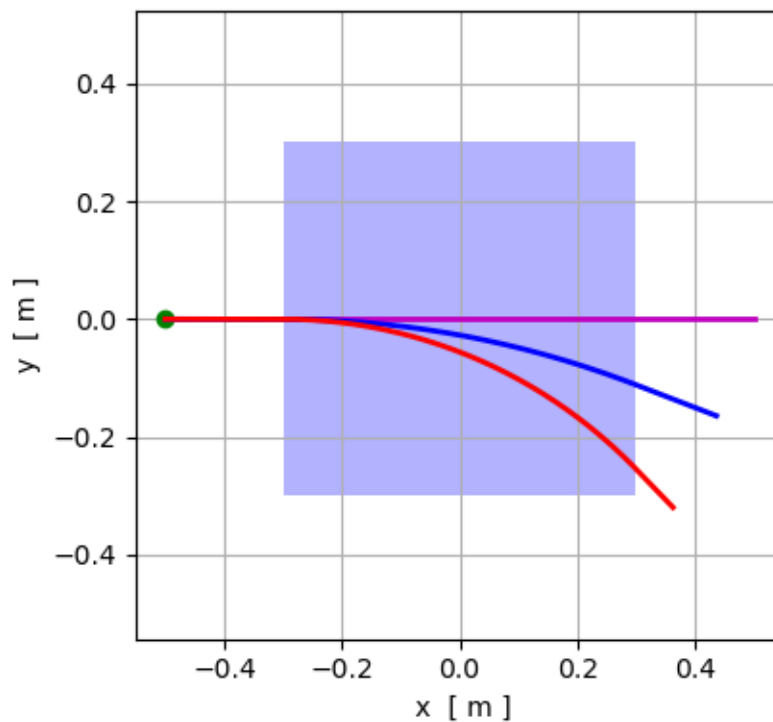


Fig. 2.3. The deflection of a **proton** and a **deuteron** in a constant magnetic field $B_z = 0.10$ T (shaded area). The stimulation time was $0.12 \mu\text{s}$.

Simulation 3 Electric field deflection `plasma006.py`

A positively charge particle in an electric field experiences an electrical force acting in the same direction as the electric field. Therefore, a positively charged particle entering an electric field will be accelerated in the direction of the electric field. We will consider the simulation of a proton initially travelling in the +X direction that enters a uniform electric field which his directed in the +Y direction.

Console Summary

INPUTS

$$m = 1.67\text{e-}27 \text{ kg}$$

$$q = 1.60\text{e-}19 \text{ kg}$$

$$x0 = -0.50 \text{ m} \quad y0 = 0.00 \text{ m} \quad z0 = 0.00 \text{ m}$$

$$vx0 = 8.0\text{e+}06 \text{ m/s} \quad vy0 = 0.0\text{e+}00 \text{ m/s} \quad vz0 = 0.0\text{e+}00 \text{ m/s}$$

$$Ex = 0.00 \text{ N/C} \quad Ey = 1.00\text{e+}06 \text{ N/C} \quad Ez = 0.00 \text{ N/C}$$

$$Bx = 0.00 \text{ T} \quad By = 0.00 \text{ T} \quad Bz = 0.00 \text{ T}$$

$$\text{Simulation time } t2 = 1.20\text{e-}07 \text{ s}$$

$$\text{Time steps } N = 10000$$

OUTPUTS

$$Ey \sim 0: ay = 9.6\text{e+}13 \text{ m/s}^2$$

$$\text{Execution time} = 0.22 \text{ s}$$

Graphical predictions

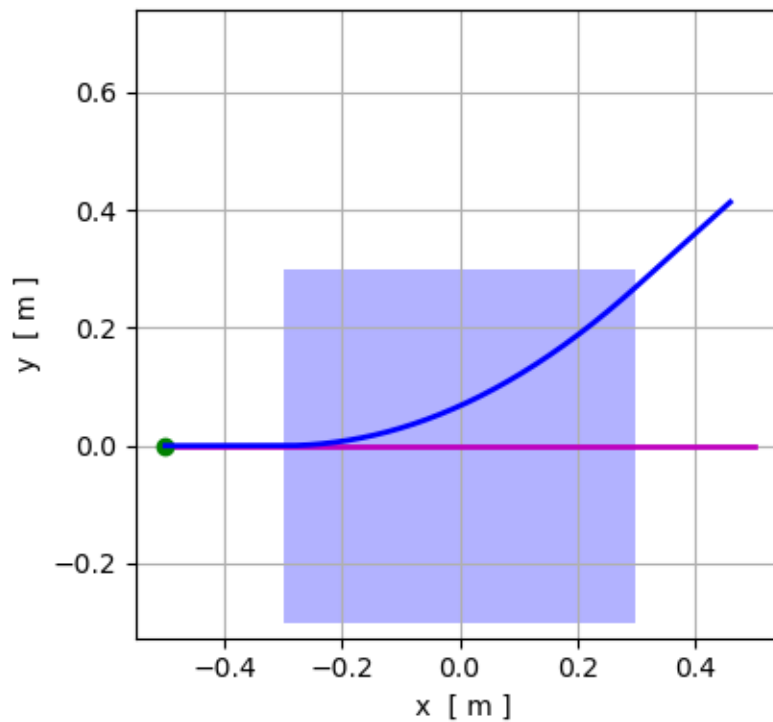


Fig. 3.1. Trajectory of a proton in the electric field. The shaded area indicates the region where the electric field is non-zero.

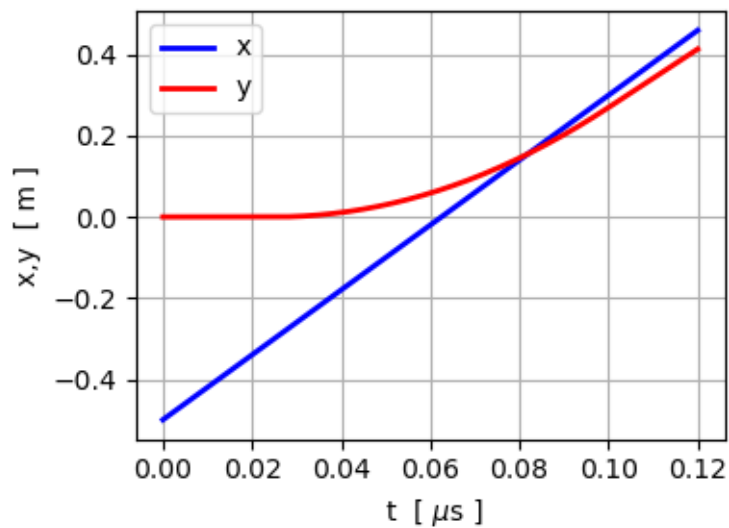


Fig. 3.2. Displacement-time graph for the motion of the proton in the electric field.

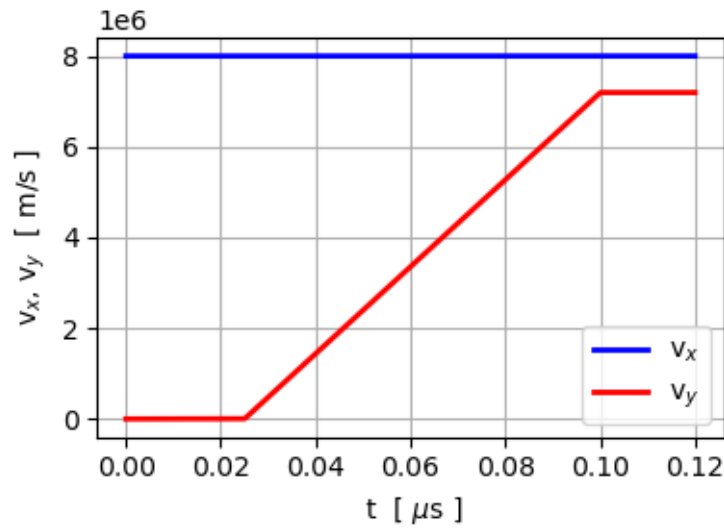


Fig. 3.3. Velocity-time graph for the motion of the proton in the electric field.

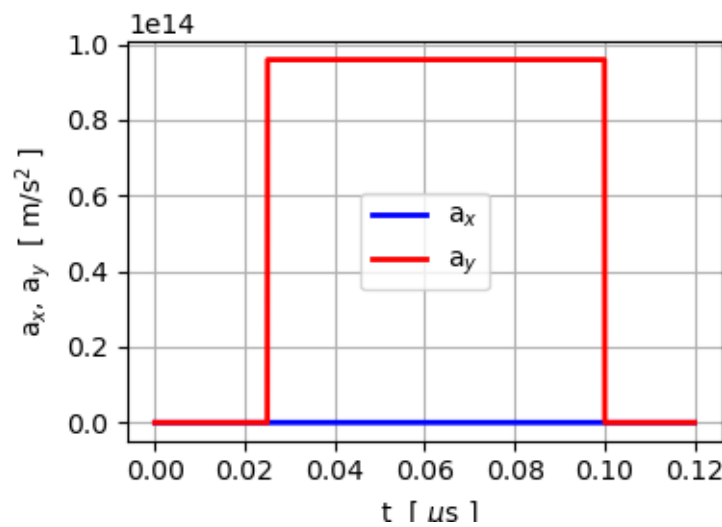


Fig. 3.4. Acceleration-time graph for the motion of the proton in the electric field.

When the electric field is zero, the velocity of the proton is constant. The x-component is zero, and the y component of the acceleration is a constant when the proton is acted upon by the electric field. The motion of the charged particle is similar to a projectile in a gravitational field.

Simulation 4 `plasma008.py`

Motion in uniform crossed magnetic and electric fields

Input parameters for a proton

$$v_{x0} = 8 \times 10^6 \text{ m.s}^{-1}$$

$$x_0 = -0.5 \text{ m}$$

$$B_z = 0.2 \text{ T}$$

$$t_2 = 0.12 \times 10^{-6} \text{ s}$$

When the magnetic force balances the electric force, the charged particle moves with constant velocity

Magnetic force = electric force

$$q v_{x0} B_z = q E_y$$

$$E_y = v_{x0} B$$

$$E_y = (8.0 \times 10^6)(0.2) \text{ V.m}^{-1} = 1.60 \times 10^6 \text{ V.m}^{-1}$$

The theoretical prediction agrees with the result of the numerical simulation.

Figure 4.1 shows the trajectories of the proton in a uniform magnetic field $B_z = 0.2 \text{ T}$ and varying electric field strengths E_y .

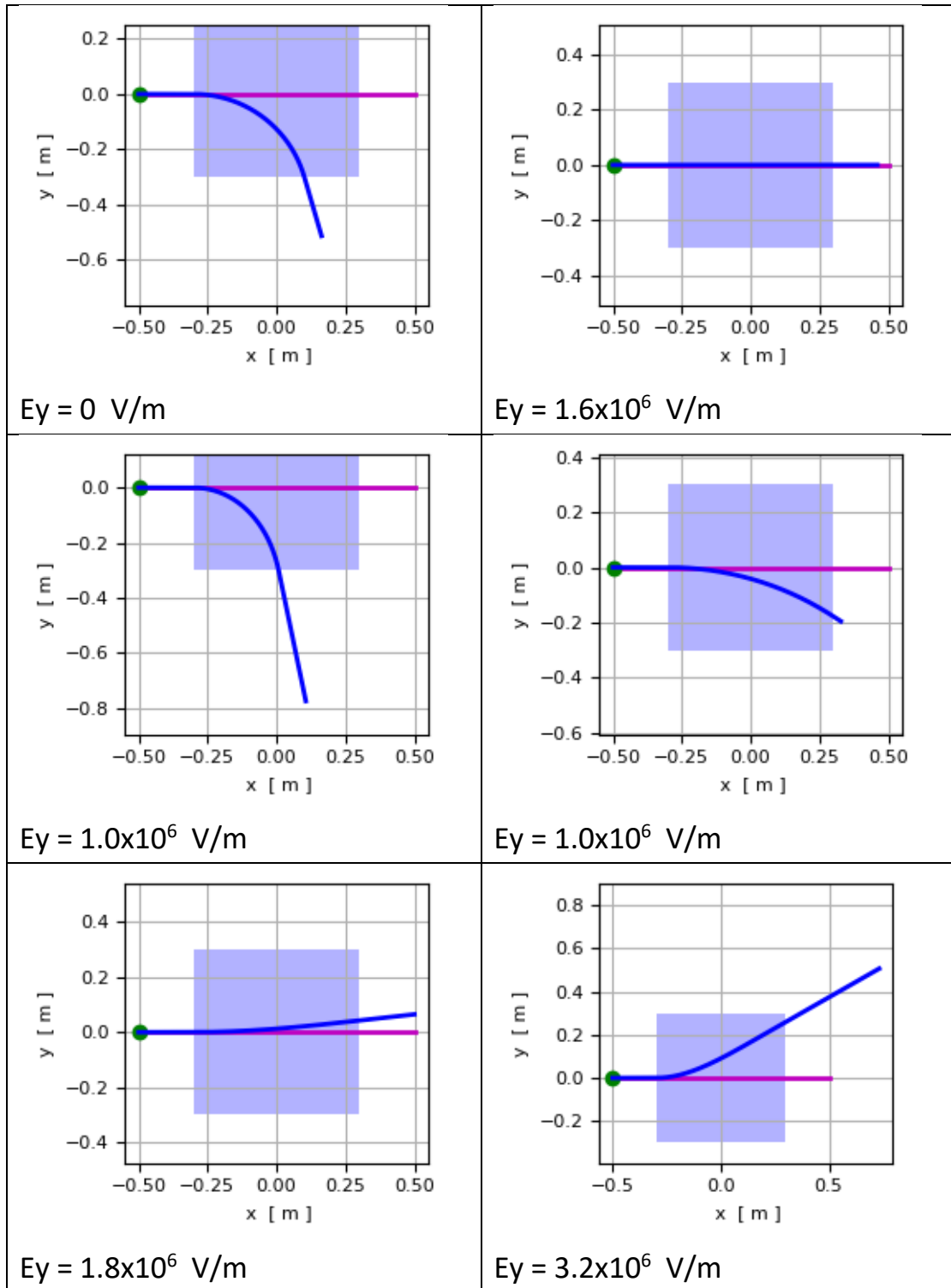


Fig. 4.1. The trajectories of the proton in a uniform magnetic field $B = 0.2$ T and varying electric field strengths.

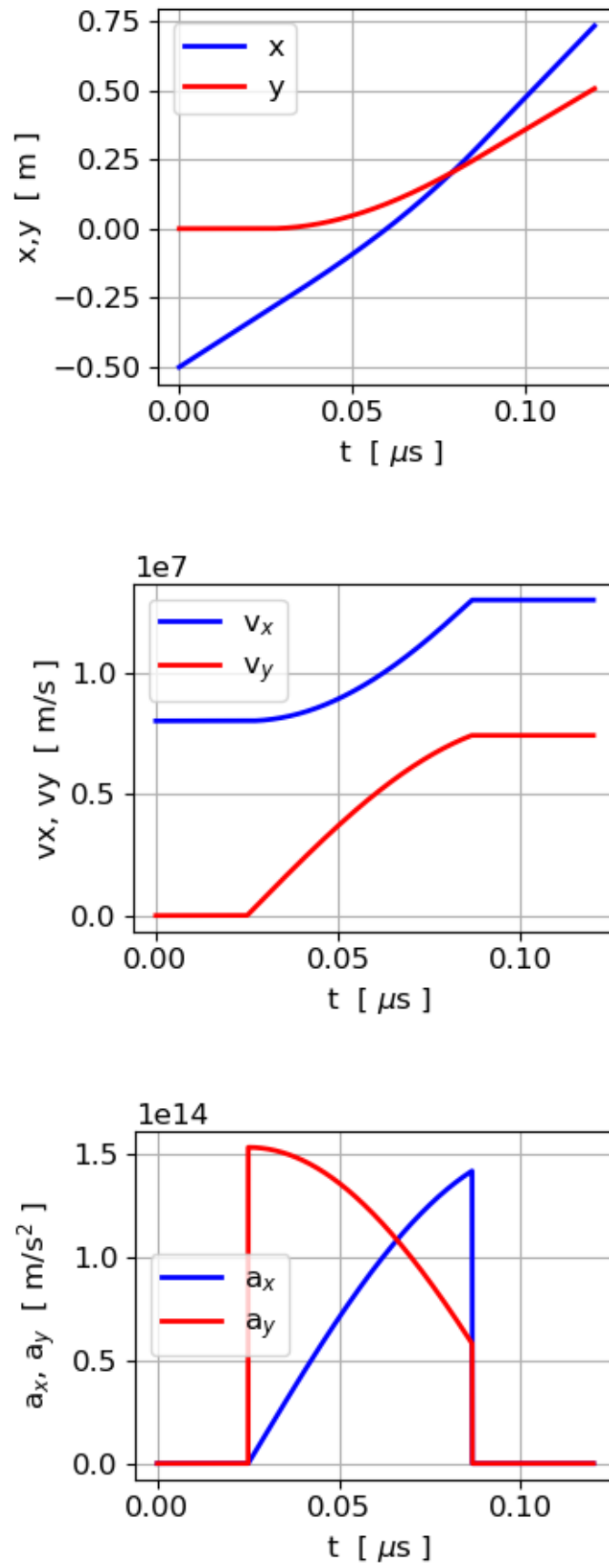


Fig. 4.2. Time evolution plots: $B_z = 0.2 \text{ T}$, $E_y = 3.2 \times 10^6 \text{ V.m}^{-1}$.

When the proton initially traveling in the +X-direction enters the region of crossed electric field and magnetic field it is acted upon by the electric force deflecting the proton in the +Y direction and the magnetic force deflecting the particle in the -Y-direction. In this case where $B_z = 0.2 \text{ T}$, $E_y = 3.2 \times 10^6 \text{ V.m}^{-1}$ the electric force is greater than the magnetic force acting on the proton. Thus, the proton is deflected in the +Y-direction and it acquires a velocity component in the +Y-direction. This results in a positive acceleration of the proton in the X-direction. The x component of the velocity is no longer constant but increases.

$$\vec{a} = (q / m) (\vec{E} + \vec{v} \times \vec{B})$$

$$a_x = (q / m) (v_y B_z) > 0$$

Simulation 5 Cycloid Motion **plasma010.py**

Consider the motion of a charged particle in uniform magnetic and electric fields. **The magnetic field is directed in the +Z direction and the electric field is in the +Y direction.**

When a positively charged particle enters the electromagnetic field region so that it is travelling in an XY plane, the electric field accelerates the charge particle resulting in an increase in the Y component of the velocity v_y . Since the positive charged particle is moving in an XY plane, the magnetic field exerts a

force on the positive charge and the faster the charge is moving, the greater the magnetic force. The direction of the force causes the charged particle to be deflected back around towards the Y axis. When the positive particle moves against the electric force it starts slowing down. As the velocity magnitude in the XY plane decreases, magnetic force decreases and the electric force takes over until the charged particle's Y component of velocity comes zero, $v_y = 0$. The positively charged particle is then accelerated again in the Y direction and the cycle is repeated, giving the **cycloid motion**. The period T for the cycloid motion is

$$T = \frac{2\pi m}{qB}$$

Input parameters

$$B_z = 0.8$$

$$E_y = 2.5e6$$

$$x_0 = -0.4; y_0 = -0.1; z_0 = 0.0$$

$$v_{x0} = 8e5; v_{y0} = 8e5; v_{z0} = 0$$

$$t_2 = 0.45e-6$$

$$q = 1.602e-19$$

$$m = 1.67e-27$$

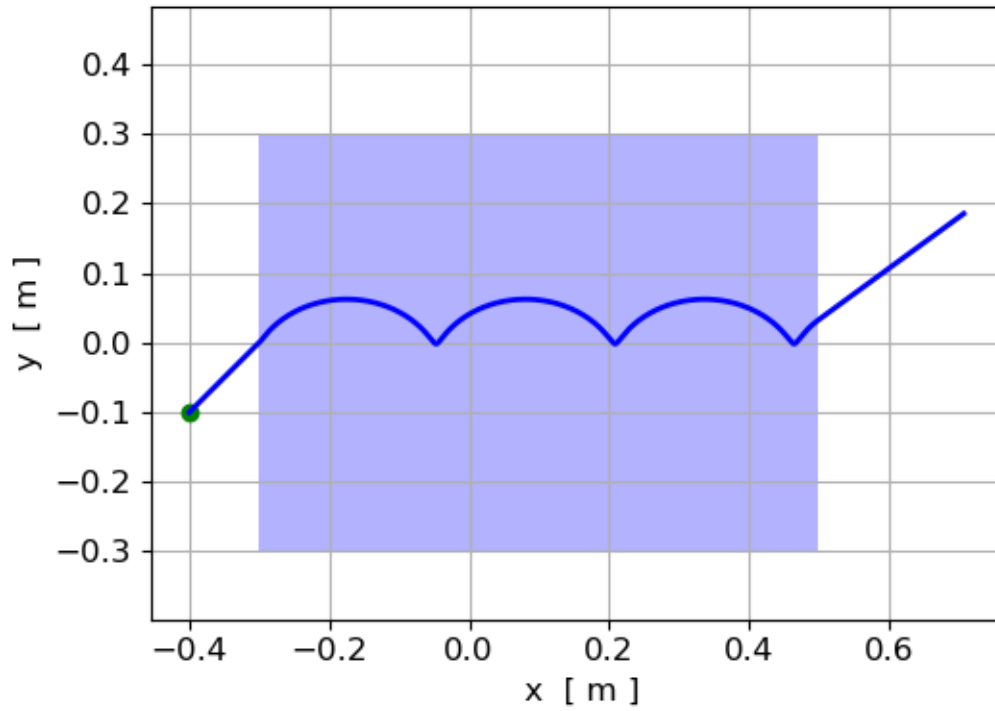


Fig. 5.1. Cycloid motion of proton in the electromagnetic field, $B_z = 0.80 \text{ T}$ and $E_y = 2.5 \times 10^6 \text{ V.m}^{-1}$.

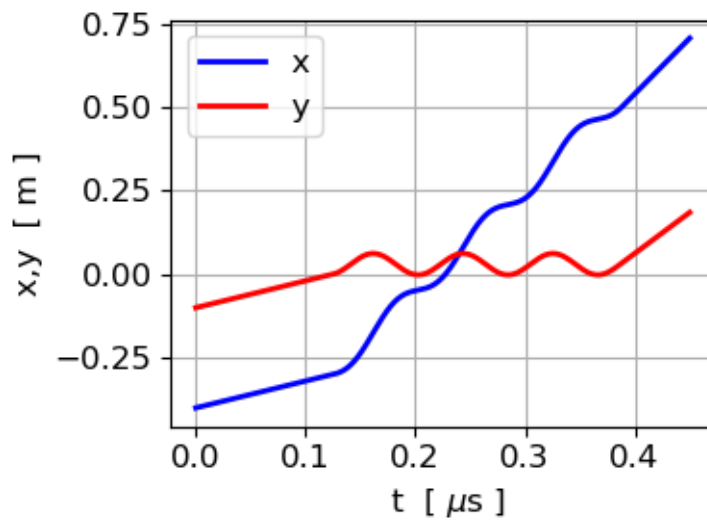


Fig. 5.2. The X and Y components of the displacement of the proton.

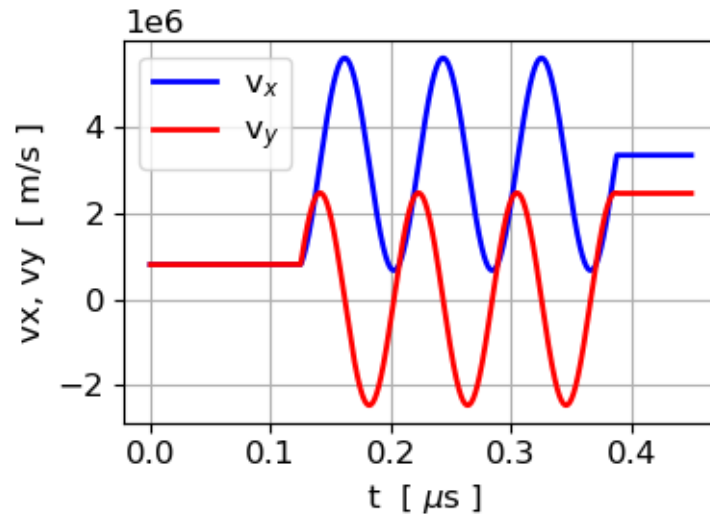


Fig. 5.3. The velocity vs time plots for the motion of the proton.

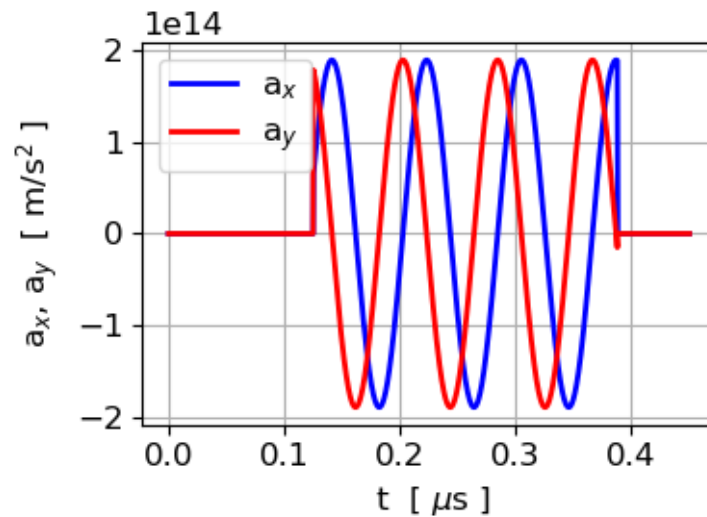


Fig. 5.4. The acceleration vs time plots for the motion of the proton.

The period for the cycloid motion is $T = 8.19 \times 10^{-8}$ s. The numerical and theoretical predictions are the same.

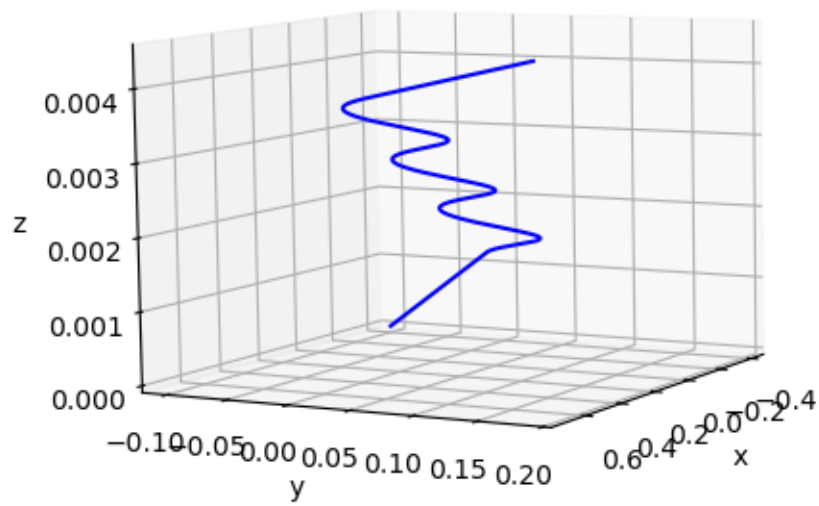


Fig. 5.5. [3D] trajectory when $v_{z0} = 1.0 \times 10^4 \text{ m.s}^{-1}$.