

DOING PHYSICS WITH PYTHON

MOVING CHARGES IN ELECTRIC AND MAGNETIC FIELDS MAGNETIC MIRROR

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INTRODUCTION

A magnetic mirror is a magnetic field configuration used to trap and reflect moving charged particles by using a weaker magnetic field in the centre and stronger magnetic fields at the ends (mirror points).

Helical Motion: Charged particles like protons, electrons or ions move in a helical (spiral) path around magnetic field lines.

Converging Field Lines: As the charged particles move toward the ends of the device, the magnetic field lines crowd closer together, thus increasing field strength towards the ends.

Conservation of magnetic moment: The particle's magnetic moment remains constant

$$\mu = m v_{\perp}^2 / 2B = \text{constant}$$

When B increases, the perpendicular velocity component $v_{\perp}$ must also increase.

Velocity Reversal: Because total kinetic energy is conserved

$$v_{\perp}^2 + v_{\parallel}^2 = \text{constant}$$

an increase in $v_{\perp}$ causes the parallel velocity $v_{\parallel}$ to drop to zero and reverse direction, bouncing the particle back toward the centre.

Magnetic Bottle: Placing two mirrors (coils) facing each other creates a magnetic bottle where the charged particles oscillate back and forth.

Plasma Confinement: Historically and currently explored in linear magnetic confinement fusion (mirror machines) to hold high-temperature plasma.

The "Loss Cone" Leakage: Particles traveling outside a specific velocity and pitch-angle range do not experience enough mirroring force and escape out the ends, making simple mirror devices inherently "leaky".

Earth's magnetic field acts as a magnetic mirror, trapping solar wind particles in the Van Allen radiation belts and producing polar auroras.



MATHEMATICAL ANALYSIS

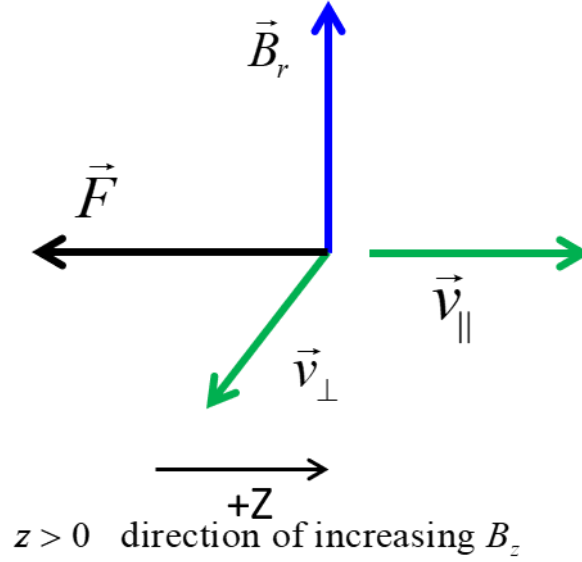
A magnetic bottle arrangement of two coils is used for the simulations of the magnetic mirror. The axis of the coil is the Z axis with the centre at $z = 0$. For this arrangement, the z component of the magnetic field B_z increases as $|z|$ increases. When $z = 0$, $B_z = B_{min}$.



For such an inhomogeneous magnetic field, a charged particle experiences a constant net Lorentz force

$$\vec{F} = q(\vec{v}_{\perp} \times \vec{B}_r) = \text{constant}$$

that has its origin in the radial component of the magnetic field. The force vector is oriented along the central field line and points in the direction of a weaker field. This force acts to decelerate and may eventually reflect a particle that has originally moved into the region of stronger field. Therefore, this field geometry is called a **magnetic mirror**.



We obtain the radial part of the magnetic field from the vanishing of the divergence of the magnetic flux density (magnetic induction)

$$\nabla \cdot \vec{B} = \frac{1}{r} \frac{\partial (r B_r)}{\partial r} + \frac{\partial B_z}{\partial z} = 0$$

When we prescribe the longitudinal gradient $\partial B_z / \partial z$ at $r = 0$, and assume this as approximately constant. Then integrating gives

$$r B_r = - \int_0^r r \frac{\partial B_z}{\partial z} dr \approx - \frac{1}{2} r^2 \left[\frac{\partial B_z}{\partial z} \right]_{r=0}$$

and obtain the radial magnetic field as

$$B_r \approx - \frac{1}{2} r \frac{\partial B_z}{\partial z}$$

Then, the net Lorentz force in Z direction, acting on the ring current with a gyroradius r_L (Larmor radius) is

$$\langle F_z \rangle \approx -\frac{1}{2} q v_{\perp} r_L \frac{\partial B_z}{\partial z}$$

This force leads to an accelerated motion of the guiding centre along the magnetic field. It slows the charged particle motion when it is moving in the Z direction of increasing magnetic field.

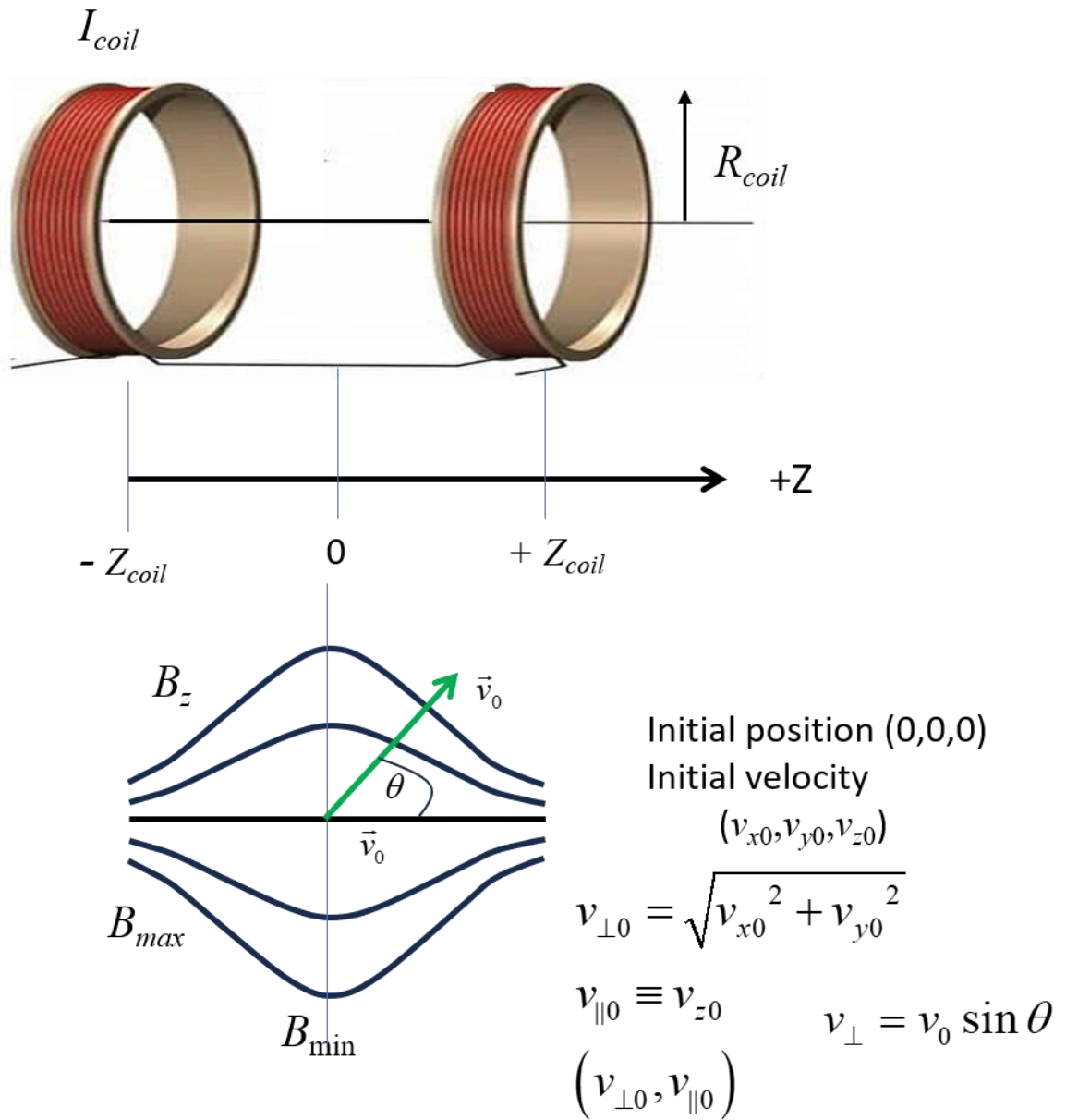


Fig. 1. A pair of circular coils can be used to make a magnetic mirror device.

When the distance between the coils is larger than their radius, the magnetic field becomes inhomogeneous along the axis of the system, as can be seen by the high density of field lines at the coil position and the lower density in the midplane. The mirror field shown in figure 1 is produced by a pair of circular currents, each of magnitude I_{coil} , at positions $z = \pm Z_{coil}$ and radius R_{coil} .

From Biot-Savart's law, the magnetic field on the axis of the two ring current arrangement is

$$B(z) = \left(\frac{\mu_0 I_{coil} R_{coil}^2}{2} \right) \left(\frac{1}{\left(R_{coil}^2 + (z - Z_{coil})^2 \right)^{3/2}} + \frac{1}{\left(R_{coil}^2 + (z + Z_{coil})^2 \right)^{3/2}} \right)$$

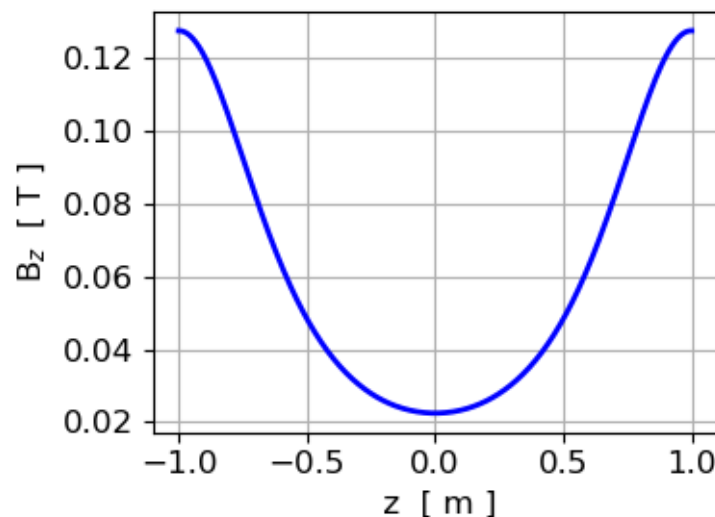


Fig. 2. Axial magnetic field B_z

$$B_{min} = 0.022 \text{ T}, \quad B_{max} = 0.127 \text{ T}$$

$$R_{coil} = 0.50 \text{ m}, \quad Z_{coil} = 1.00 \text{ m}, \quad I_{coil} = 1.00 \times 10^5 \text{ A}$$

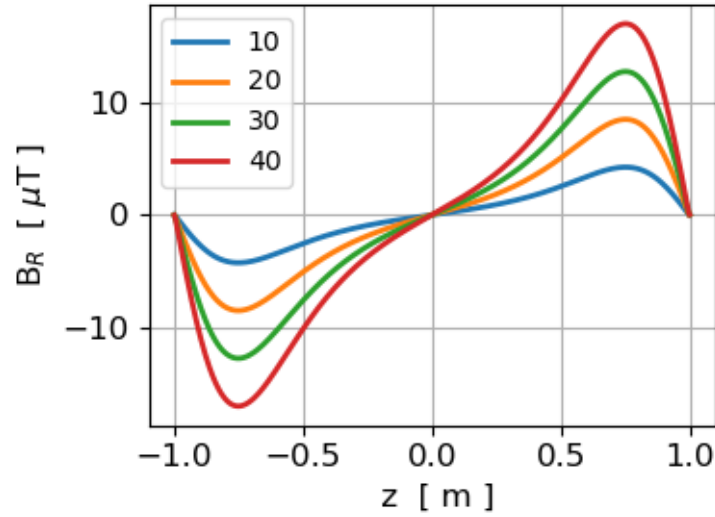


Fig. 3. Radial magnetic field B_r
Radii R [mm]: 10, 20, 30, 40
 $R_{coil} = 0.50$ m, $Z_{coil} = 1.00$ m, $I_{coil} = 1.00 \times 10^5$ A

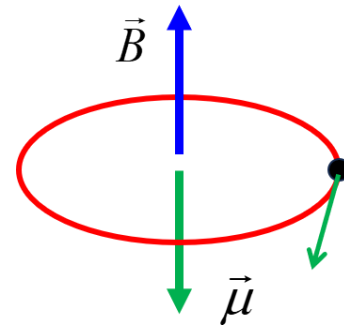
Magnetic Moment

The circular orbit of the particle about the central field line in the geometry of the magnetic bottle can be considered as an electric current. This ring current has an associated magnetic moment μ , which is the product of the current I flowing at the edge of a circular disk with Larmor radius r_L and the area A of this disk. The current is given by the charge q performing a revolution in the gyroperiod T_C (cyclotron period)

$$\mu = I A \quad I = |q| / T_C \quad A = \pi r_L^2 \quad 1 / T_C = \frac{\omega_C}{2\pi} \quad r_L = \frac{m v_{\perp}}{|q| B} \quad \omega_C = \frac{|q| B}{m}$$

$$\mu = \frac{m v_{\perp}^2}{2 B}$$

The magnetic moment is a vector that is antiparallel to the ambient magnetic field B , and according to Lenz's rule, weakens the external field.



With this definition of the magnetic moment, we can rewrite

$$\langle F_z \rangle \approx -\frac{1}{2} q v_{\perp} r_L \frac{\partial B_z}{\partial z}$$

$$\langle F_z \rangle \approx -\mu \frac{\partial B_z}{\partial z}$$

This shows that the gyrating particle experiences a force like a piece of diamagnetic matter in an inhomogeneous magnetic field (diamagnetic materials are substances that create an induced magnetic field opposite to an applied external magnetic field, resulting in a weak repulsive force). The motion of a positive ion creates a magnetic dipole that is antiparallel to the acting magnetic field. The same is true for electrons, which have the opposite charge and the opposite sense of gyration. It is shown in classical mechanics that the action integral over a periodic orbit, $\oint p dq$ is a conserved quantity of the system. This concept can be extended to weak gradients, in which the orbit is nearly periodic. The associated action integrals then are no longer strict invariants but become *adiabatic invariants*

When we assume that the diamagnetic force in $\langle F_z \rangle$ is a valid description, we obtain an energy relation for the motion of the guiding centre by multiplying with v_z

$$\begin{aligned}\langle F_z \rangle &= m \dot{v}_z = -\mu \frac{\partial B_z}{\partial z} \\ m v_z \dot{v}_z &= -\mu \frac{dz}{dt} \frac{\partial B_z}{\partial z} \\ \frac{d}{dt} \left(\frac{1}{2} m v_z^2 \right) &= -\mu \frac{dB}{dt}\end{aligned}$$

Here, dB/dt is the change in the magnetic field, which the guiding centre experiences by moving along the central field line. For the guiding centre, we have no radial magnetic field and therefore can drop the index z . A time-invariant magnetic field does not alter the kinetic energy, as can be seen from

$$\vec{F} \cdot d\vec{s} = q(\vec{v} \times \vec{B}) \cdot \vec{v} dt = 0$$

because the Lorentz force is always perpendicular to the trajectory. The change in kinetic energy can then be written as

$$\begin{aligned}\frac{d}{dt} \left(\frac{1}{2} m v_z^2 + \frac{1}{2} m v_\perp^2 \right) &= \frac{d}{dt} \left(\frac{1}{2} m v_z^2 + \mu B \right) \\ -\mu \frac{dB}{dt} + \frac{d}{dt} (\mu B) &= -\mu \frac{dB}{dt} + \mu \frac{dB}{dt} + B \frac{d\mu}{dt} = 0 \\ \frac{d\mu}{dt} &= 0\end{aligned}$$

Hence, the magnetic moment is conserved to the same degree of accuracy as the diamagnetic force gave a sufficiently accurate description of the motion of the guiding centre.

Confinement of a charged particle that moves along the central field line of magnetic bottle arrangement.

At any position, the velocity components of the particle are

$$v_{\parallel} \equiv v_z \quad \text{along the axis}$$

$$v_{\perp} = \sqrt{v_x^2 + v_y^2} \quad \text{gyromotion speed}$$

In the symmetry plane of the mirror ($z = 0$) the magnetic field is $B = B_{\min}$ and there the particle has initial velocity $(v_{\perp 0}, v_{z0})$. The maximum magnetic field $B = B_{\max}$ is found in the vicinity of the mirror coils. The motion of the particle is governed by the conservation of energy and the adiabatic invariance of the magnetic moment:

Conservation of energy

$$\frac{1}{2} m v_{z0}^2 + \frac{1}{2} m v_{\perp 0}^2 = \frac{1}{2} m v_z^2 + \frac{1}{2} m v_{\perp}^2$$

$$v_{z0}^2 + v_{\perp 0}^2 = v_z^2 + v_{\perp}^2$$

Adiabatic invariance of the magnetic moment

$$\mu = \frac{m v_{\perp 0}^2}{2 B_{\min}} = \frac{m v_{\perp}^2}{2 B} = \text{constant}$$

$$\frac{v_{\perp 0}^2}{B_{\min}} = \frac{v_{\perp}^2}{B} = \text{constant}$$

When the particle moves into regions of higher magnetic field, its parallel energy is consumed by the diamagnetic force. At the same time, the gyrofrequency increases, which leads to a larger kinetic energy of the gyromotion

$$v_{\perp} \uparrow \Rightarrow v_z \downarrow \quad v_{\perp} \downarrow \Rightarrow v_z \uparrow$$

A reflection by the magnetic mirror occurs when the energy of parallel motion becomes zero at any position before B attains its maximum value $B_{\max}$. When $v_z = 0$ and setting $v_{\perp 0} = v_0 \sin \theta$ gives

$$v_z^2 = v_0^2 \left(1 - \frac{B_{\max}}{B_{\min}} \sin^2 \theta \right) = 0$$

This means that the stopping point of a particle is only dependent on the starting angle θ with respect to the magnetic field. It is independent of the magnitude of the initial velocity v_0 . All particles with a starting angle $\theta > \theta_m$ are confined, while the particles with $\theta < \theta_m$ can overcome the mirror point $B_{\max}$ and form the *loss cone* in velocity space. The angle of the loss cone is

$$\theta_m = \sin^{-1} \left(\sqrt{\frac{B_{\min}}{B_{\max}}} \right)$$

The quantity $R_m = B_{\max}/B_{\min}$ is called the *mirror ratio*, which defines the confinement quality of a mirror device. A large mirror ratio is equivalent to a small loss-cone angle.

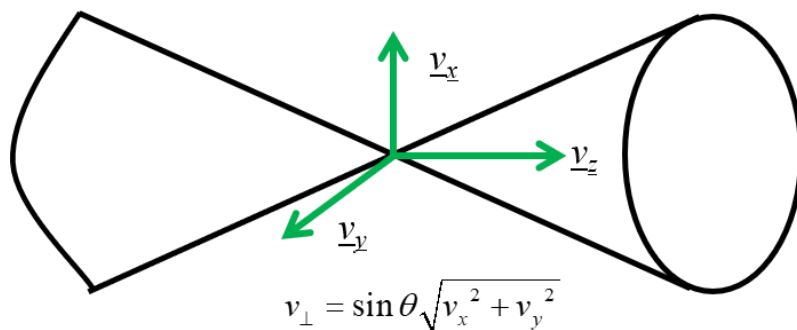


Fig. 4. A loss cone in velocity space. Particles whose velocity vectors lie inside the cone are not reflected by the magnetic field.

If plasma is placed inside a magnetic mirror device, then all of the particles whose velocities lie in the loss cone promptly escape, but the remaining particles are confined. Unfortunately, that is not the end of the story. There is no such thing as an absolutely collisionless plasma. Collisions take place at a low rate, even in very hot plasmas. One important effect of collisions is to cause diffusion of particles in velocity space. Thus, collisions in a mirror device continuously scatter trapped particles into the loss cone, giving rise to a slow leakage of plasma out of the device. Even worse, plasmas whose distribution functions deviate strongly from an isotropic Maxwellian (for instance, a plasma confined in a mirror device) are prone to *velocity-space instabilities* that tend to relax the distribution function back to a Maxwellian. Such instabilities can have a disastrous effect on plasma confinement in a mirror device.

SIMULATIONS

Default input parameters

$$\text{proton: } m = 1.673 \times 10^{-27} \text{ kg, } q = + 1.602 \times 10^{-19} \text{ C}$$

$$R_{coil} = 0.50 \text{ m, } Z_{coil} = 1.00 \text{ m, } I_{coil} = 1.00 \times 10^5 \text{ A}$$

Output parameters

$$B_{min} = 0.022 \text{ T, } B_{max} = 0.127 \text{ T}$$

Loss cone angle

$$\theta_m = \sin^{-1} \left(\sqrt{\frac{B_{min}}{B_{max}}} \right) = 24.83^\circ$$

Simulation 1

Simulation time $t_S = 150 \text{ } \mu\text{s}$

Loss cone angle = 24.83°

Initial velocity:

$$v_{x0} = 3.50 \times 10^4 \text{ m.s}^{-1} \quad v_{y0} = 3.50 \times 10^4 \text{ m.s}^{-1} \quad v_{z0} = 1.00 \times 10^5 \text{ m.s}^{-1}$$

$$\text{radial velocity } v_{r0} = 4.95 \times 10^4 \text{ m.s}^{-1}$$

Launch angle $\theta_0 = 26.33^\circ$

max $z = 0.85 \text{ m}$

$\theta_0 > \theta_m \Rightarrow$ proton trapped and oscillates back and forth between the mirrors (coils)

Fig. 5. The proton oscillates with simple harmonic motion between the two mirrors. The amplitude of the oscillation is 0.85 m.

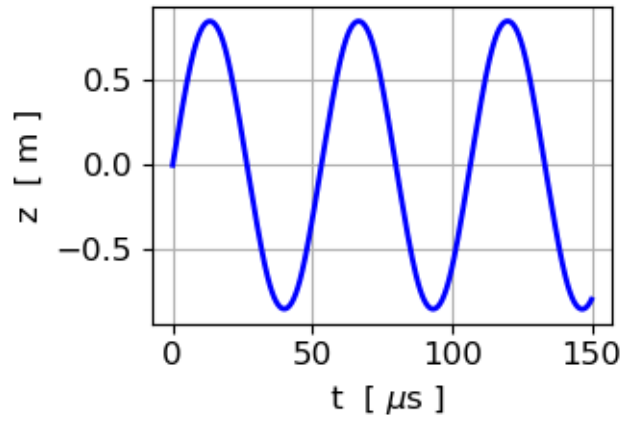


Fig. 6. Energy is conserved as the velocity of the proton remains constant. As the radial velocity increases the parallel component of the velocity decreases. At the extremes of the motion the proton stops and reverses direction.

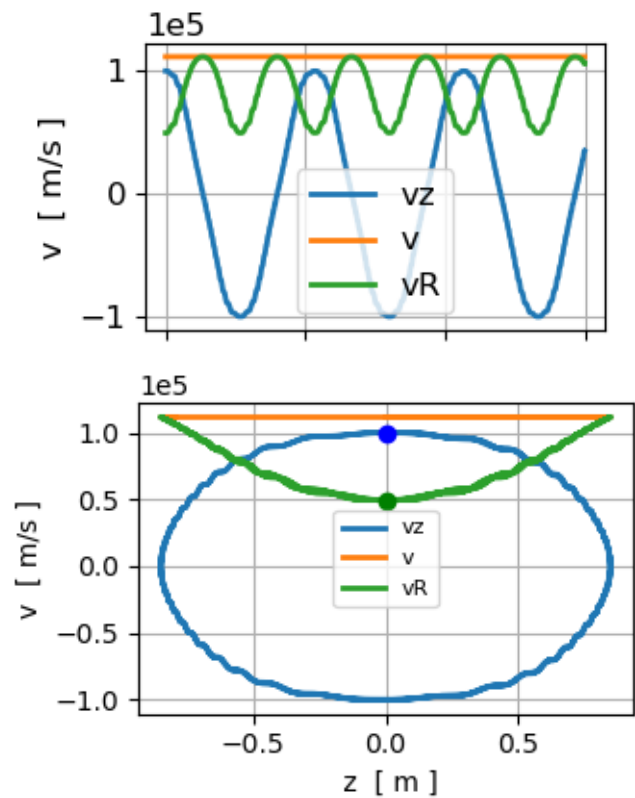


Fig. 7. The radial displacement spirals outward then inward. The minimum radius occurs near the mirrors.

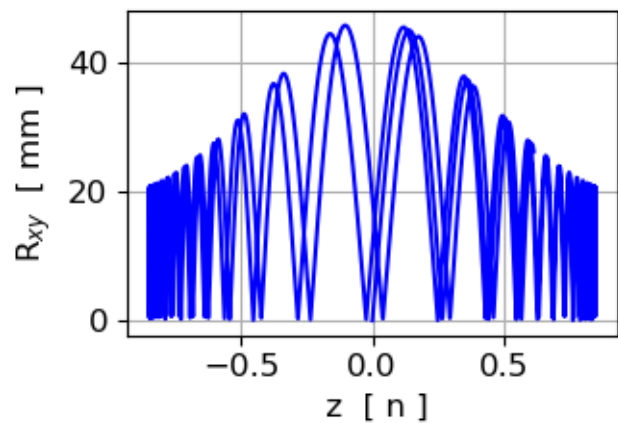


Fig. 8. x, y plot of the spiral motion of the proton.

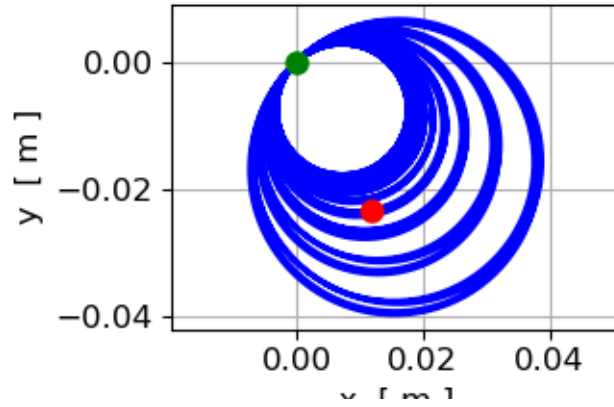


Fig. 9. Radial displacements.

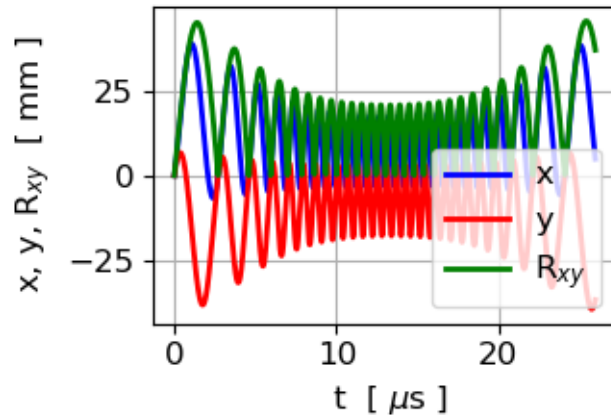
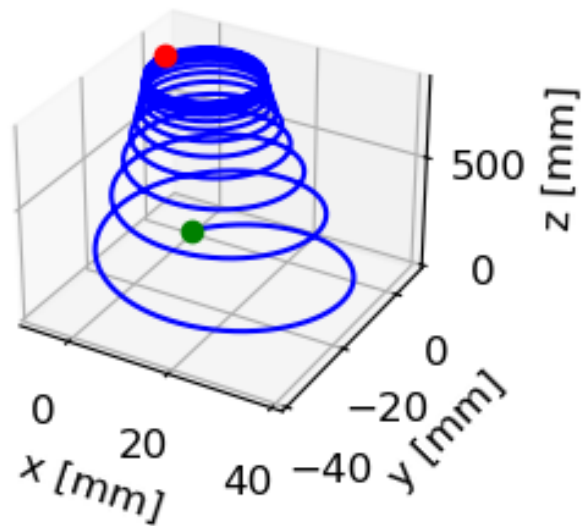


Fig. 10. [3D] trajectory of the proton for $\frac{1}{4}$ cycle.



Initial velocity:

$$v_{x0} = 8.50 \times 10^4 \text{ m.s}^{-1} \quad v_{y0} = 3.50 \times 10^4 \text{ m.s}^{-1} \quad v_{z0} = 1.00 \times 10^5 \text{ m.s}^{-1}$$

$$\text{radial velocity } v_{r0} = 9.19 \times 10^4 \text{ m.s}^{-1}$$

Launch angle $\theta_0 = 42.59^\circ$

max $z = 0.50 \text{ m}$

When the launch angle is increased, the radial magnetic force on the moving charged particle produces a large force opposing its motion towards the mirrors resulting in it being reflected at a greater distance from each mirror.

Simulation 2

The launch angle acts as a bifurcation parameter. Slight changes in the launch angle near the value of the loss cone angle can make a dramatic difference to the trajectory of the charged particle.

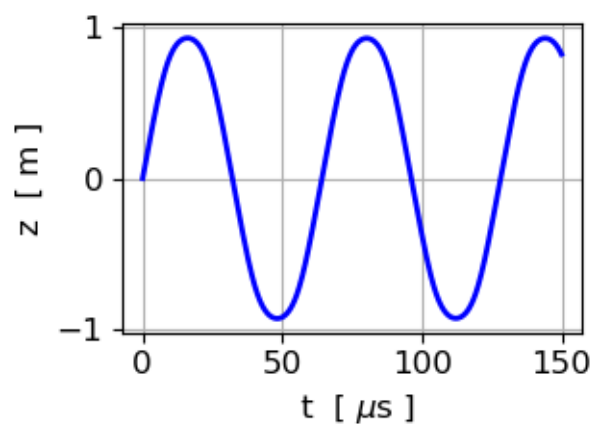
Initial velocity:

$$v_{x0} = 3.30 \times 10^4 \text{ m.s}^{-1} \quad v_{y0} = 3.30 \times 10^4 \text{ m.s}^{-1} \quad v_{z0} = 1.00 \times 10^5 \text{ m.s}^{-1}$$

$$\text{radial velocity } v_{r0} = 4.67 \times 10^4 \text{ m.s}^{-1}$$

Launch angle $\theta_0 = 25.02^\circ > \theta_m = 24.83^\circ$

Fig. 12. The proton is trapped in the magnetic bottle since $\theta_0 = 25.02^\circ > \theta_m = 24.83^\circ$. as $\theta_0 \rightarrow \theta_m$ from above, then $|z| \rightarrow 1$.



Initial velocity:

$$v_{x0} = 3.20 \times 10^4 \text{ m.s}^{-1} \quad v_{y0} = 3.20 \times 10^4 \text{ m.s}^{-1} \quad v_{z0} = 1.00 \times 10^5 \text{ m.s}^{-1}$$

$$\text{radial velocity } v_{r0} = 4.53 \times 10^4 \text{ m.s}^{-1}$$

Launch angle $\theta_0 = 24.35^\circ < \theta_m = 24.83^\circ$

The charged particle is no longer trapped by the magnetic bottle device since $\theta_0 = 24.35^\circ < \theta_m = 24.83^\circ$.

Fig. 11. The proton escapes the magnetic device when $\theta_0 < \theta_m$

