# DOING PHYSICS WITH PYTHON

## MODELLING A MASS - SPRING SYSTEM

## Free oscillations, Damping, Force oscillations (impulsive and sinusoidal)

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## DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive

GitHub

**sim001.py  sim001A.py (impulsive driving force)**

The Codes use the Python function **solve_ivp()** to solve the equation of motion for a mass-spring system. The displacement, velocity, acceleration, forces and energies are computed.

**sim001B.py**

Plot of the response curve (amplitude $A$ vs driving frequency $f_D$ / $f_0$) for the oscillations of a mass - spring system.

# INTRODUCTION

Vibration is the periodic, back-and-forth oscillatory motion of an object or particle around a central equilibrium point.

- **Equilibrium Point:** The resting position where forces on the object are balanced. When disturbed, a restoring force pulls the object back toward this point.
- **Amplitude:** The maximum distance the object moves away from its equilibrium point.
- **Period:** The time for the one complete oscillation of the object.
- **Frequency:** The number of complete back-and-forth cycles an object finishes in one second.
- **Energy Transfer:** Vibrations constantly convert energy back and forth between **potential energy** (stored in a spring or elastic bond) and **kinetic energy** (motion of the mass).
- **Damping:** The gradual loss of vibration energy due to friction or air resistance, which eventually causes the motion to stop.
- **Free Vibration:** Occurs when a system is disturbed and allowed to move on its own without external influence.
- **Forced Vibration:** Occurs when an external force repeatedly drives a system to keep it moving.

- **Resonance:** Occurs when an external driving frequency matches the natural frequency of an object, causing a massive increase in the amplitude of the vibration.

The equation of motion of motion of the mass – spring system is

(1) $$m\frac{d^2x(t)}{dt^2} = -k\,x(t) - b\,\frac{dx(t)}{dt} + F_D(t)$$

and can be expressed as

(2) $$\ddot{x} = -\left(\frac{k}{m}\right)x - \left(\frac{b}{m}\right)\dot{x} + \left(\frac{1}{m}\right)F_D$$

$t$ time [s]

$\ddot{x}$ acceleration [m.s$^{-2}$]

$\dot{x}$ , $v$ velocity [m.s$^{-1}$]

$x$ displacement [m] equilibrium point $x$ = 0

$m$ system mass [kg]

$k$ spring constant [N.m$^{-1}$]

$b$ damping constant (damping coefficient) [kg.s$^{-1}$]

$F_D$ external driving force [N]

$x(0), v(0)$ initial conditions

$K, U, E$ kinetic energy, potential energy, total energy [J]

Equation (2) is solved using the Python function **solve_ivp()**.

## SIMULATIONS

The default parameters are set so that the period of oscillation is $T_0$ = 1.00 s.

$$m = 0.506606 \text{ kg} \quad k = 20.00 \text{ N.m}^{-1}$$
$$\Rightarrow \quad f_0 = 1.000 \text{ Hz} \quad T_0 = 1.000 \text{ s}$$

A Fourier transform of the displacement $x$ is performed and the power spectral density $psd$ as a function of frequency $f$ is plotted.

### Simulation 1 Free oscillations: Simple harmonic motion

**sim001.py**

For the free oscillations leading to simple harmonic motion, the resonance or natural frequency is

$$(3) \qquad \omega_0 = \sqrt{\frac{k}{m}} \qquad f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \omega_0 = 2\pi f_0$$

The system is given an initial displacement and the subsequent motion computed.

SHM $b$ = 0, $F_D$ = 0

The displacement, velocity and acceleration of the object all oscillate sinusoidally with time as shown in figure 1.1. The total energy of the system remains constant. The orbit in phase space is an ellipse.

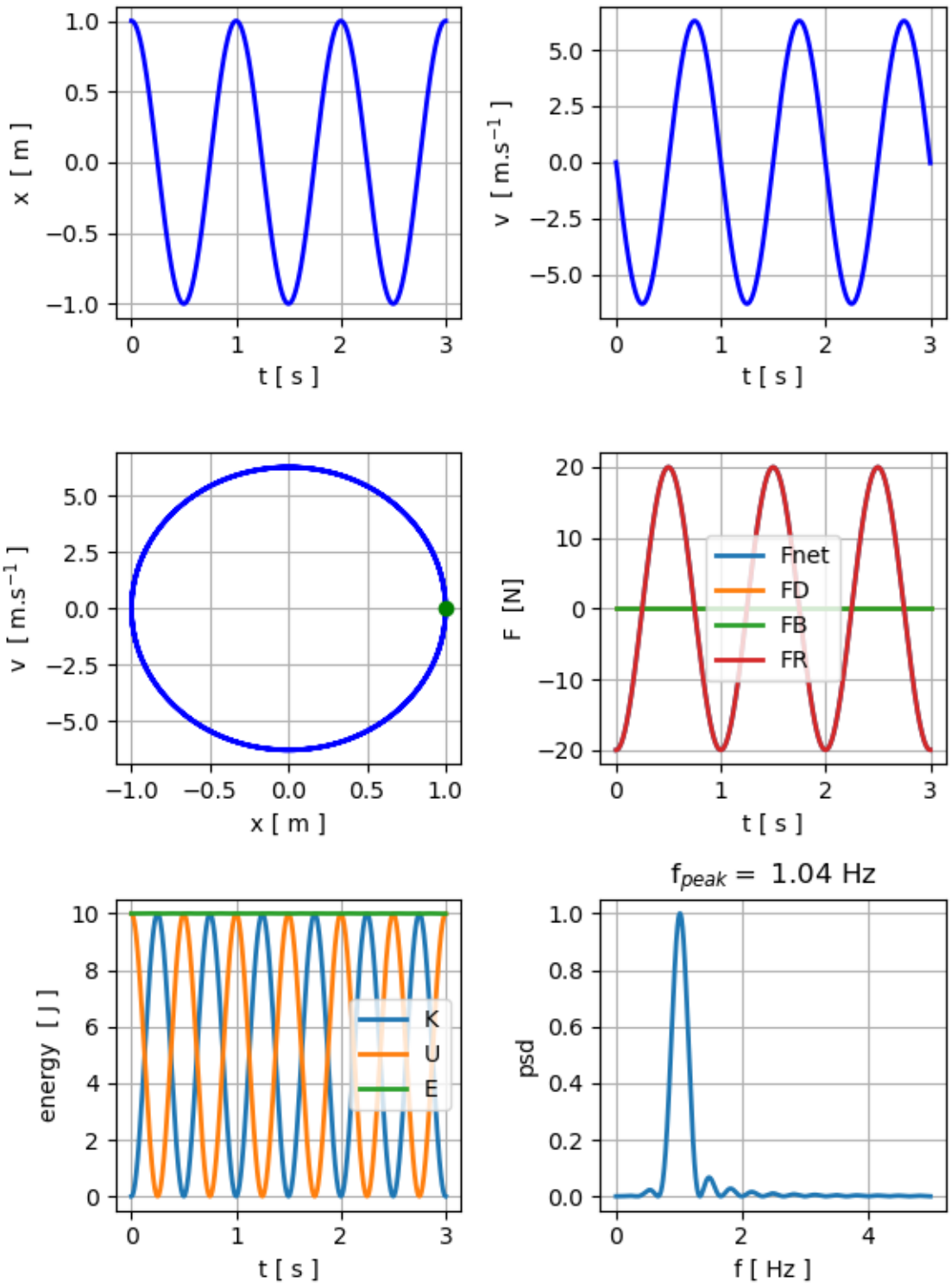



Fig. 1.1. Simple harmonic motion. Only the restoring $F_R$ force is non-zero.

## Simulation 2 Free oscillations: Damped motion sim001.py

Initially we discussed the case of ideal simple harmonic motion where the total energy remained constant and the displacement followed a sine curve, for an infinite time interval. In practice, some energy is always dissipated by a resistive or viscous process. The presence of the damping term will always result in energy loss. For example, the amplitude of a freely swinging pendulum will always decay with time as energy is lost. The presence of resistance to motion means that another force is active, which is taken as being proportional to the velocity. The frictional force acts in the direction opposite to that of the velocity Newton's Second law becomes

$$(4) \qquad m\frac{d^2x(t)}{dt^2} = -k\,x(t) - b\,\frac{dx(t)}{dt}$$

The motion of the object is highly dependent upon the value of the damping coefficient, $b$. For small values of $b$, then the restoring force is the dominant force acting on the object. For large values of $b$, the damping force dominates and all oscillations disappear if $b$ is large enough.

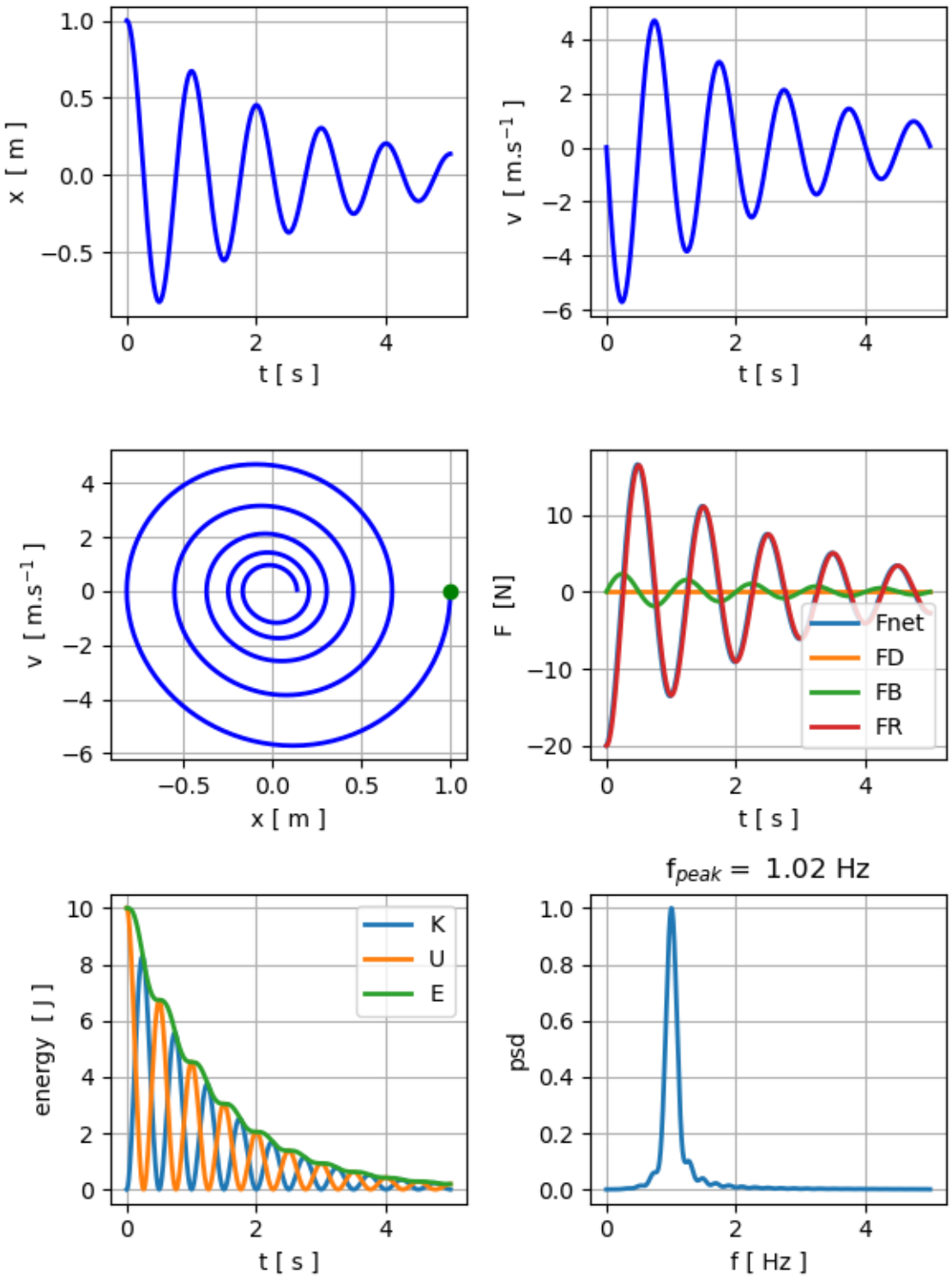



Fig. 2.1. $b$ = 0.40 kg.s$^{-1}$ The system is lightly damped and gives oscillatory **damped simple harmonic motion**. The frequency of the decaying oscillations is $f$ = 1.0 Hz.

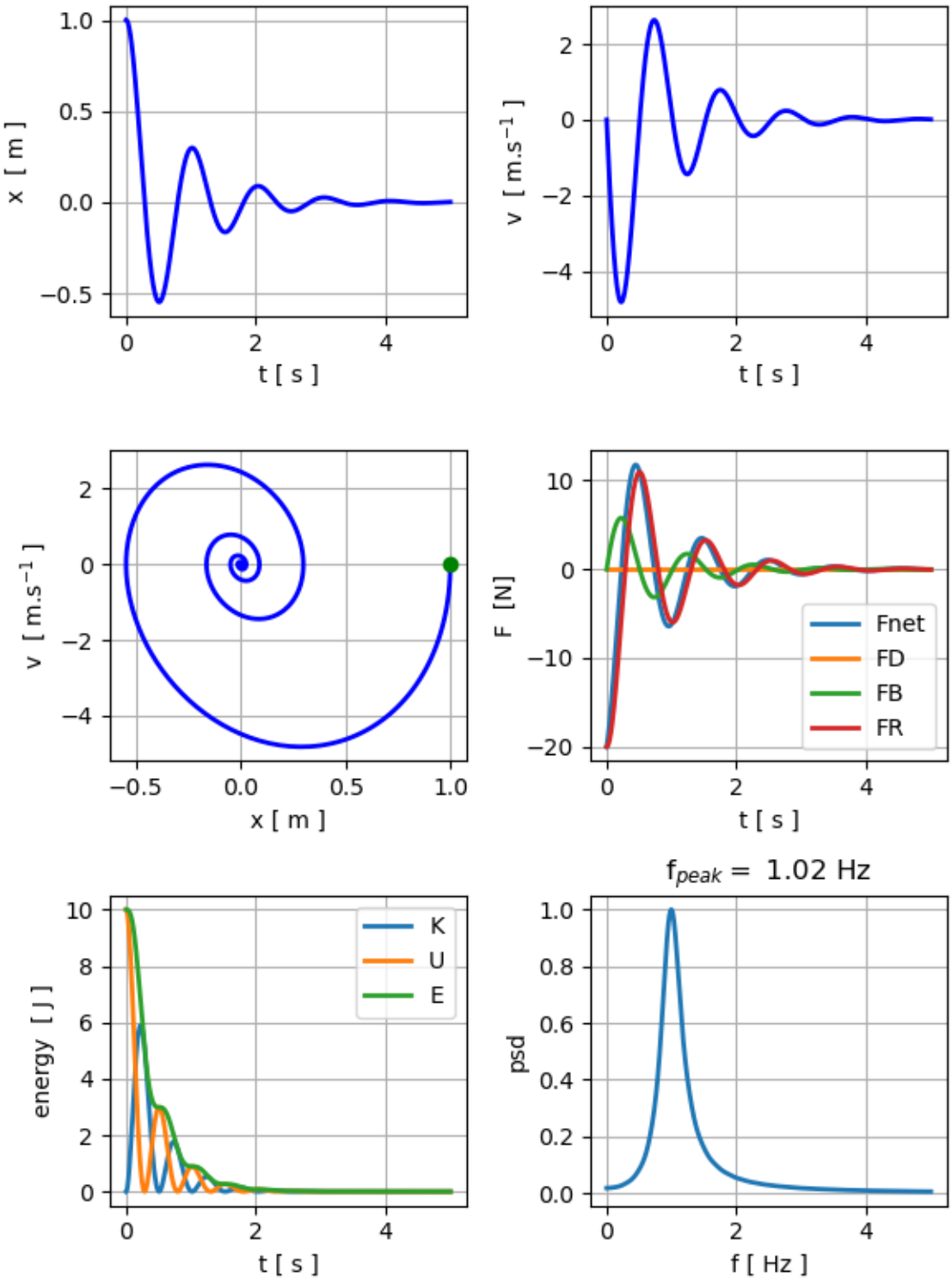



Fig. 2.2. $b$ = 1.20 kg.s$^{-1}$ The system is under damped and gives oscillatory **damped simple harmonic motion**. The frequency of the decaying oscillations is $f$ = 1.0 Hz. The oscillations die away after a few cycles.

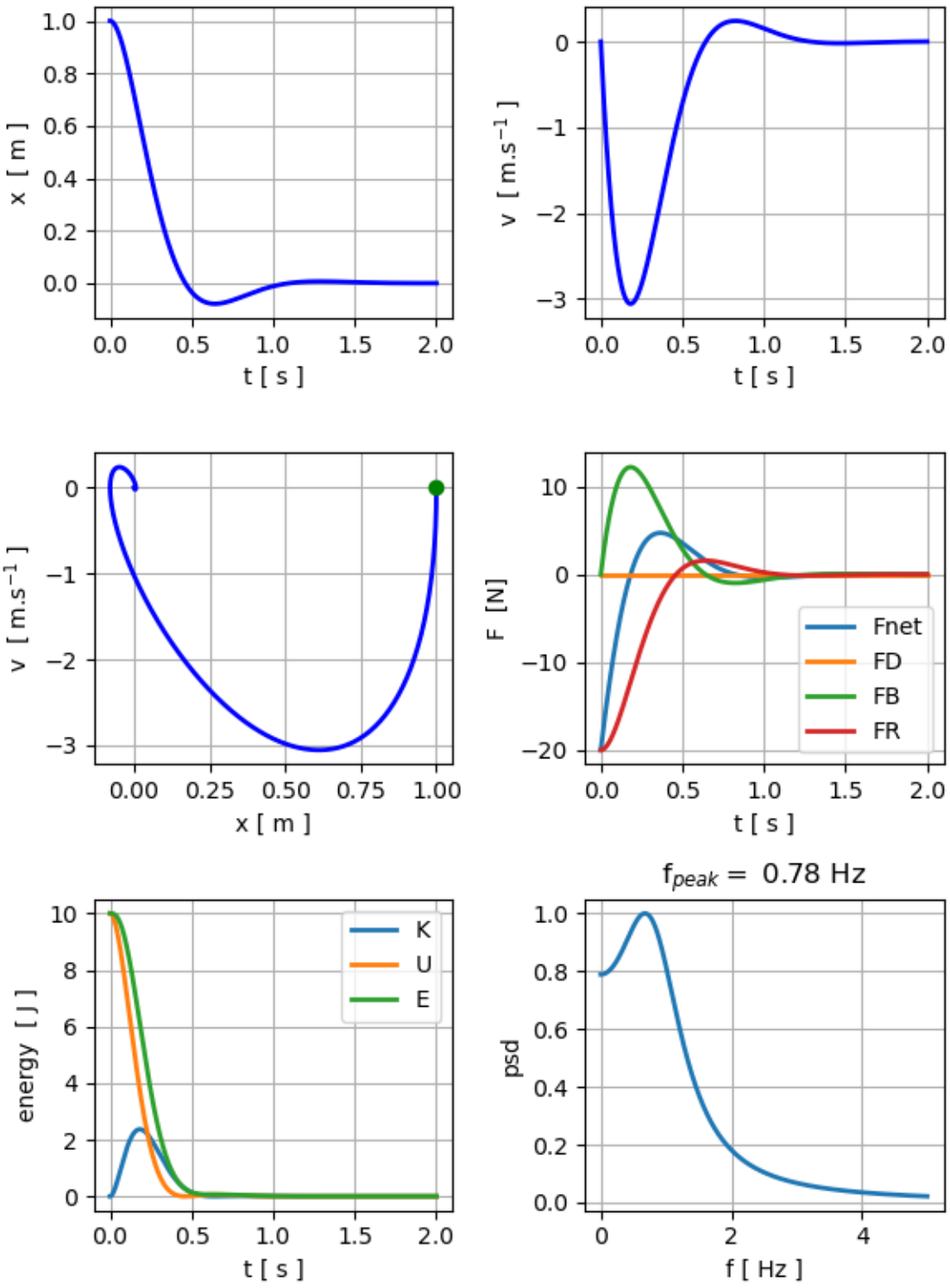



Fig. 2.3. $b$ = 4.0 kg.s$^{-1}$ The system is strongly damped with zero complete cycles.

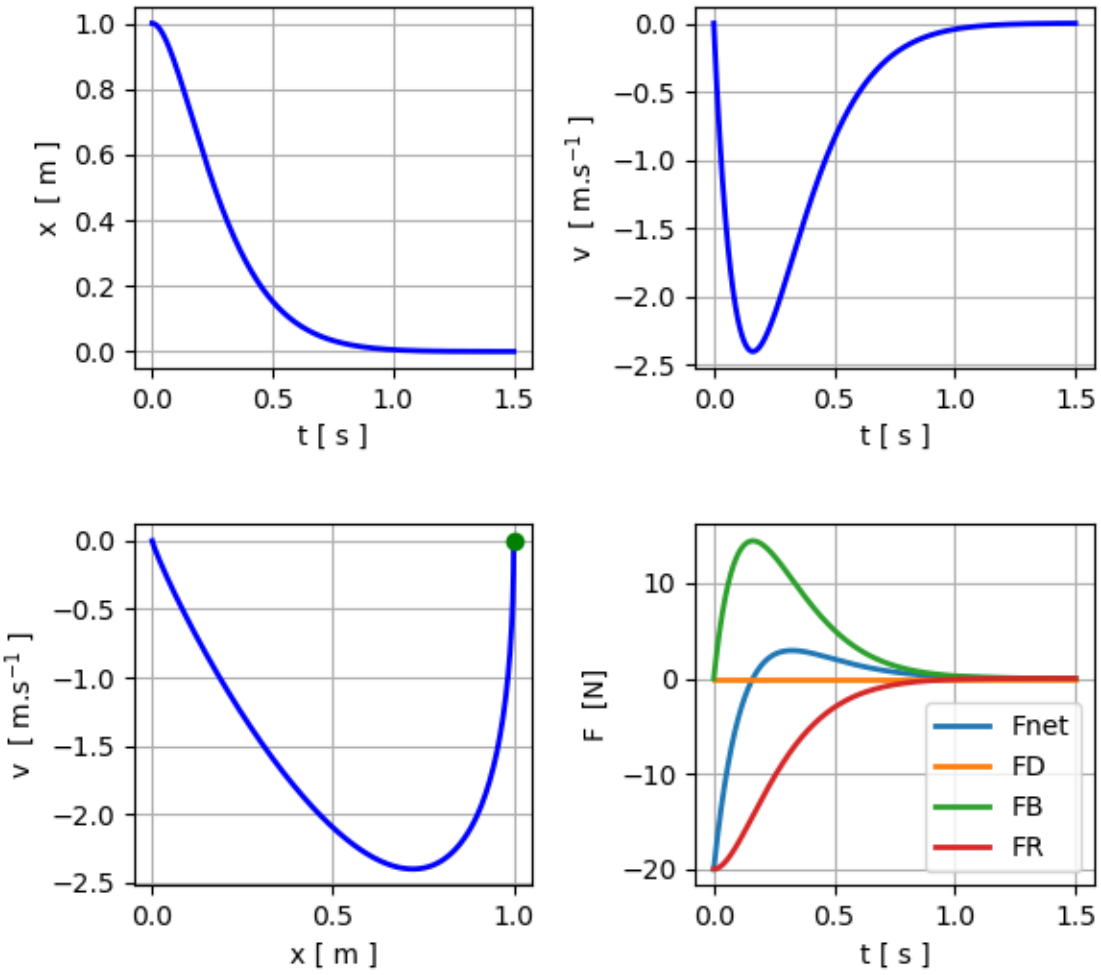



Fig. 2.4. $b$ = 6.0 kg.s$^{-1}$ The system is **critically damped** and the system returns to its equilibrium position, $x$ = 0 without the displacement becoming negative in the shortest possible time interval (1.2 s).

Critical damping is where a displaced physical system returns to its static equilibrium position in the shortest possible time without oscillating or overshooting. For values of the damping coefficient greater than the critical value, it takes longer for the system to return to the equilibrium position. For example, car shock absorbers are tuned near critical damping to absorb bumps quickly without letting the car bounce repeatedly

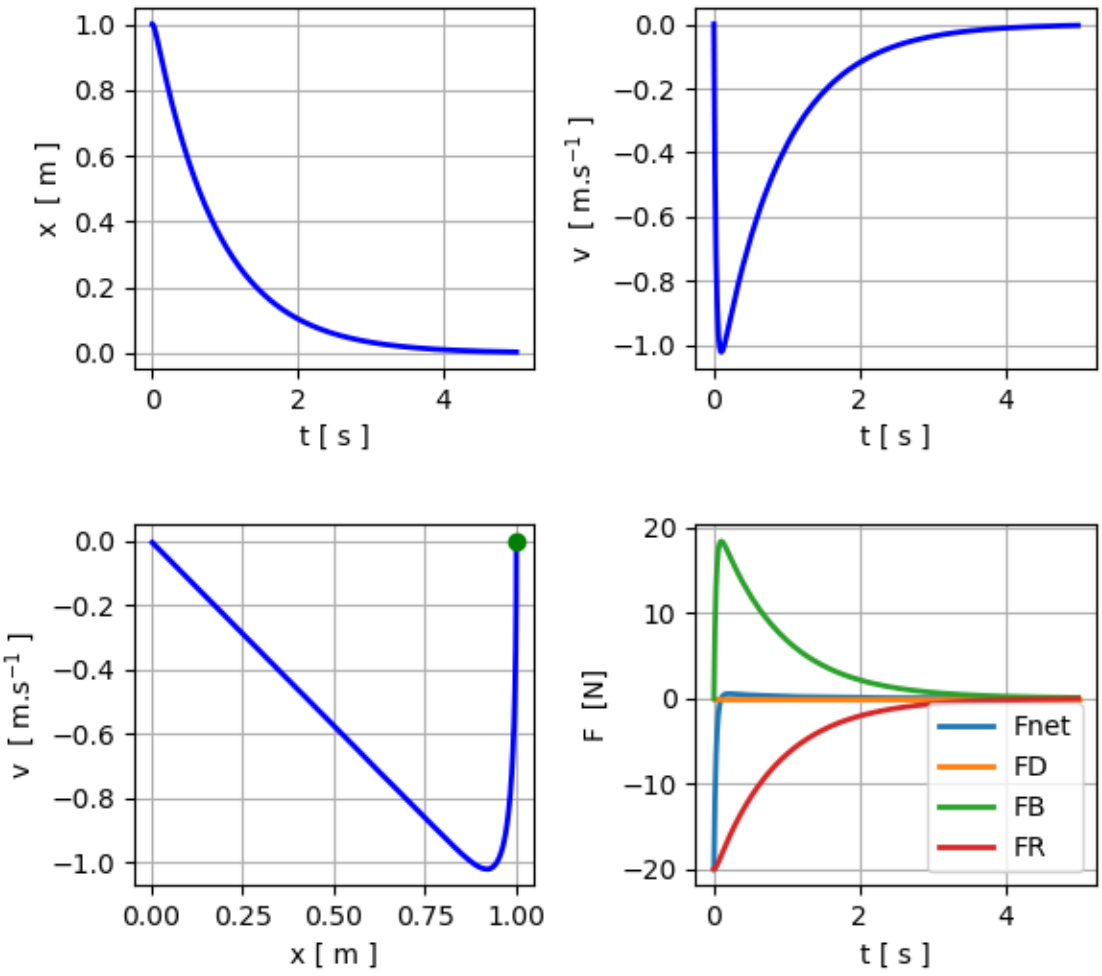



Fig. 2.5. $b$ = 18.0 kg.s$^{-1}$ The system is **over damped** and the system can take a long time to returns to its equilibrium position, $x$ = 0. When $b$ = 18.0 kg.s$^{-1}$, it takes more than 5.0 s for the object to reach the equilibrium position.

- **Underdamped $b < 6$** The system oscillates back and forth with decaying amplitude before settling.
- **Critically Damped $b = 6$** The system reaches equilibrium in the fastest time with zero bounce or oscillation.
- **Overdamped $b > 6$** The system returns to equilibrium very slowly due to excessive resistance or drag

### Simulation 3 Impulsive driving force sim001A.py

When the mass-spring system is subjected to a short, constant impulsive force, the system experiences a sudden transfer of momentum that changes the velocity to the mass. The system's response is that it will still oscillate at its natural frequency of vibration about an equilibrium position which is determined by the applied impulsive driving force.

For a system in which there is zero damping, the equation of motion to be solved is

$$(5) \qquad m\frac{d^2x(t)}{dt^2} = -k\,x(t) + F_D(t)$$

$$t_{start} \le t \le t_{stop} \quad F_D(t) = S\,(\text{constant}), \quad \text{otherwise } F_D(t) = 0$$

where $S$ is the strength of the driving force.

The impulsive driving force results in a new equilibrium position at $x = x_e$. Therefore, while the impulsive force acts, the oscillations will be centered around the new equilibrium position at $x = x_e$ and not $x$ = 0. The potential energy is thus given by

(6) $$U = \tfrac{1}{2} k \left( x - x_e \right)^2$$

The response of the system depends upon the times at which the impulsive force starts to act and stops. When the system is first acted upon by the impulsive force, it may increase or decrease the velocity (kinetic energy) of the mass.

For the simulations using the Code **sim001A.py** the impulsive force acts for the time interval from $t$ = 3.0 s to $t$ = 6.0 s. The strength of the impulsive force is S = +10 N or – 10 N. The natural frequency and period are: $f_0$ = 1.00 Hz and $T_0$ = 1.00 s.

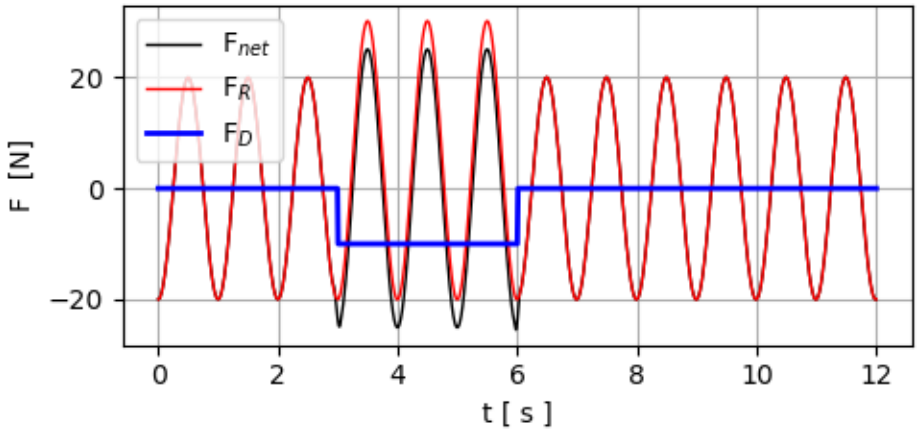



Fig. 3.1. The strength of the impulsive force is $S$ = - 10 N.

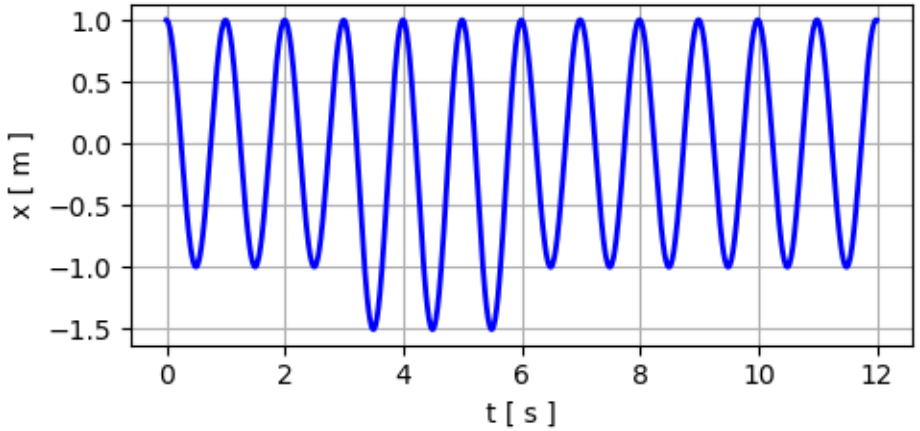



Fig. 3.2. When the impulsive force acts, $x_e$ = + 0.25 m.

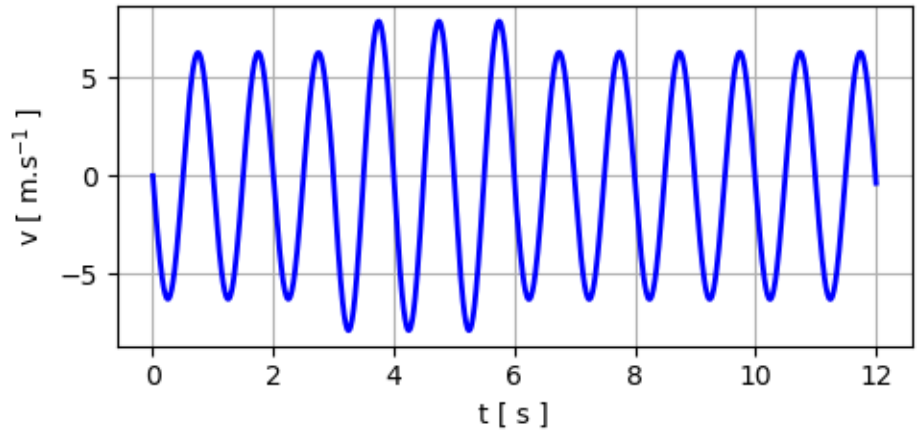



Fig. 3.3. When the impulsive force acts, the amplitudes of the displacement and velocity increase.

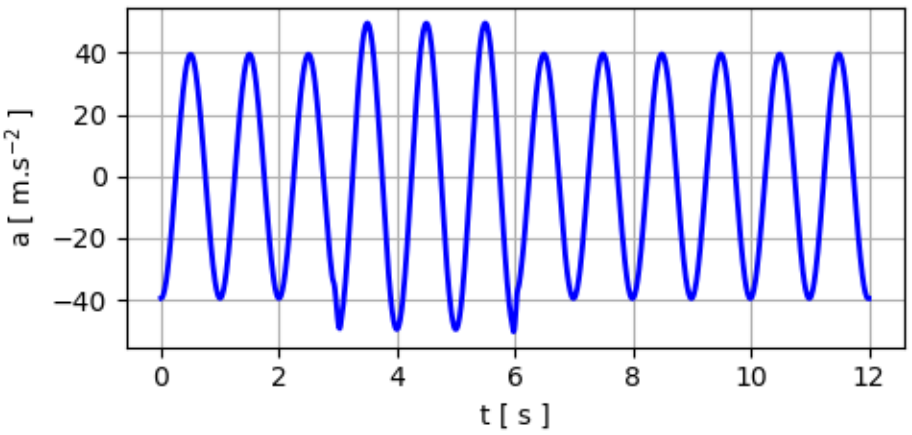



Fig. 3.3. When the impulsive force acts, the amplitude of the acceleration increases. The frequency and period of the oscillation match the natural frequency and period: $f$ = 1.0 Hz and $T$ = 1.0 s.

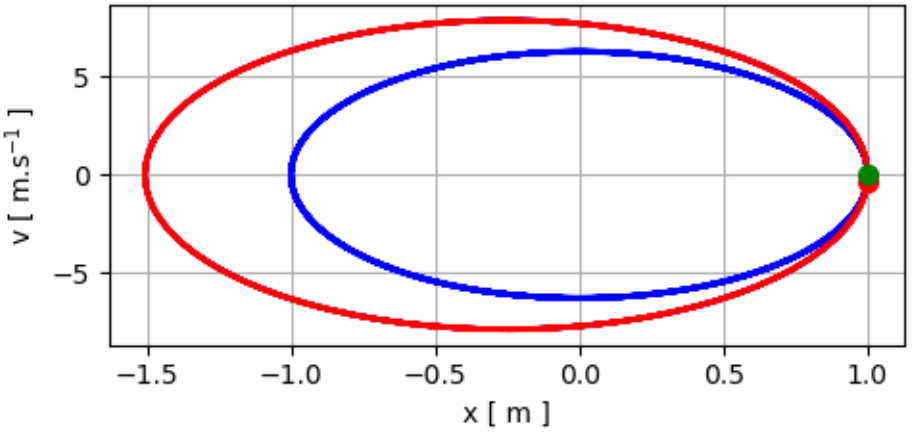



Fig. 3.4. The red curve is the orbit in phase space when the impulsive force acts.

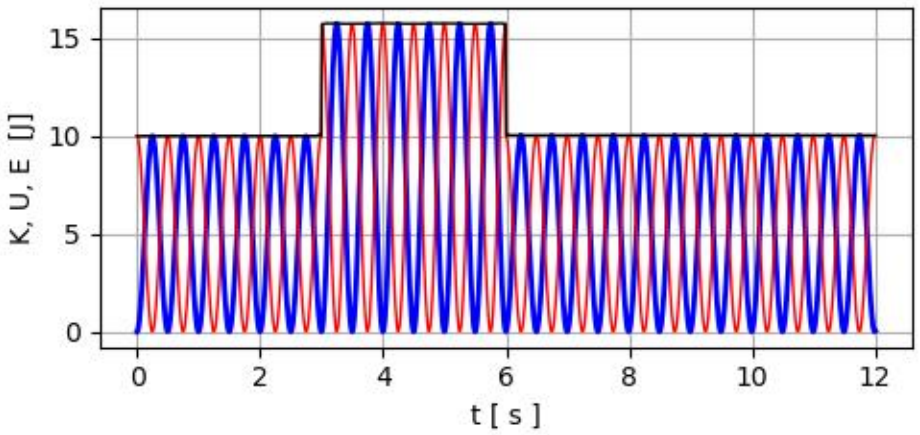



Fig. 3.5. The total energy is a conserved quantity. When the impulsive force acts, work is done on the system to increase the total energy of the system.

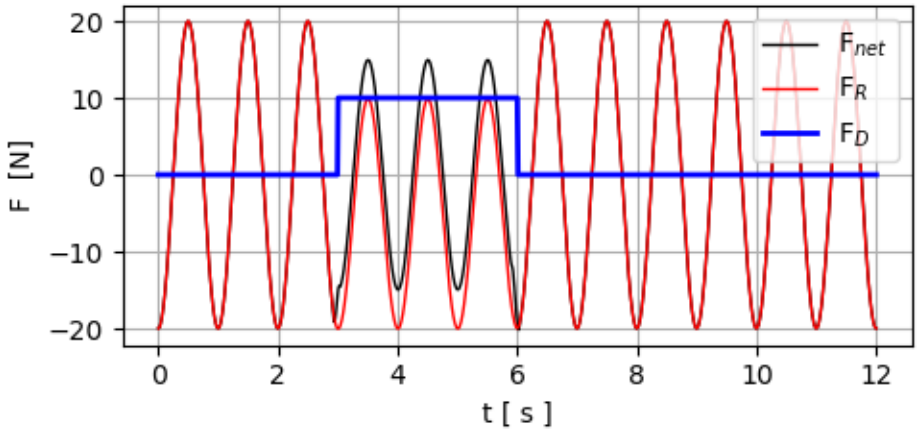



Fig. 3.6. The strength of the impulsive force is $S$ = + 10 N.

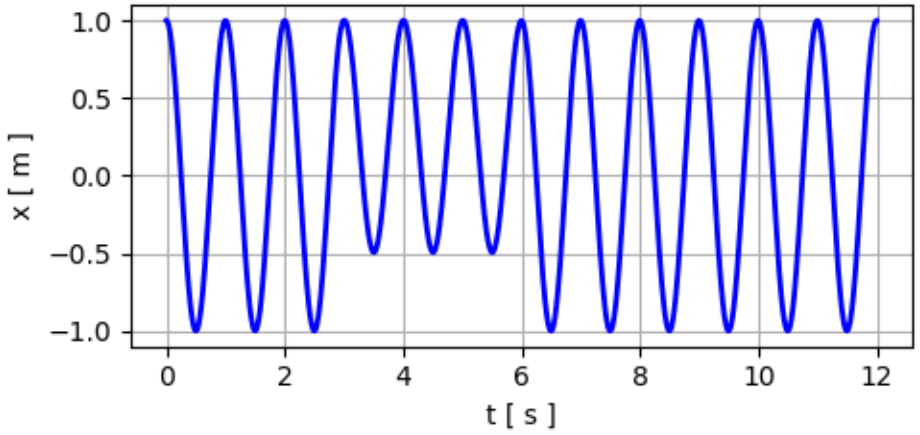



Fig. 3.7. When the impulsive force acts, $x_e$ = + 0.25 m.

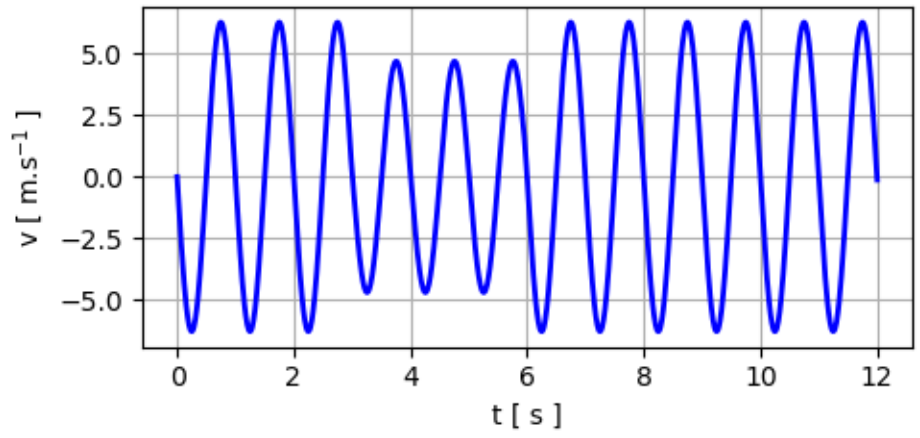



Fig. 3.8. When the impulsive force acts, the amplitudes of the displacement and velocity decrease.

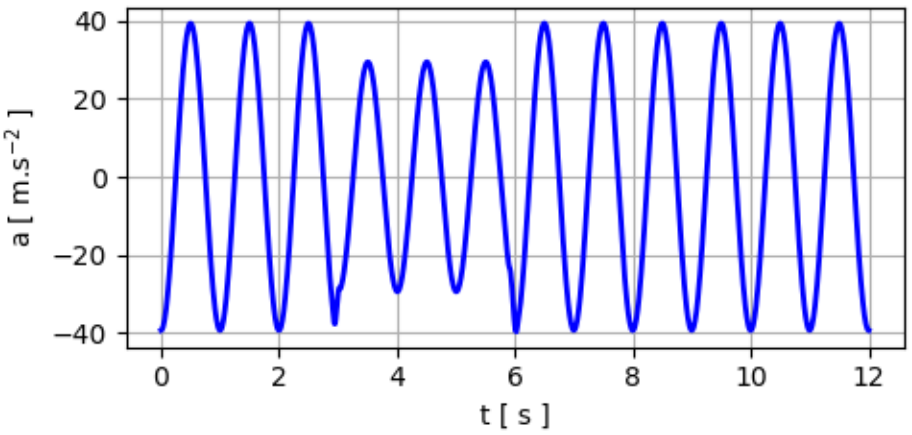



Fig. 3.9. When the impulsive force acts, the amplitude of the acceleration decreases. The frequency and period of the oscillation match the natural frequency and period: $f$ = 1.0 Hz and $T$ = 1.0 s.

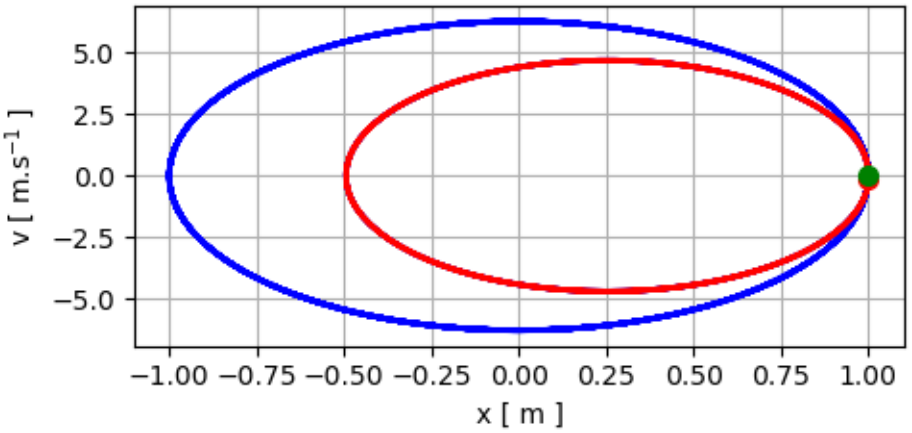



Fig. 3.4. The red curve is the orbit in phase space when the impulsive force acts.

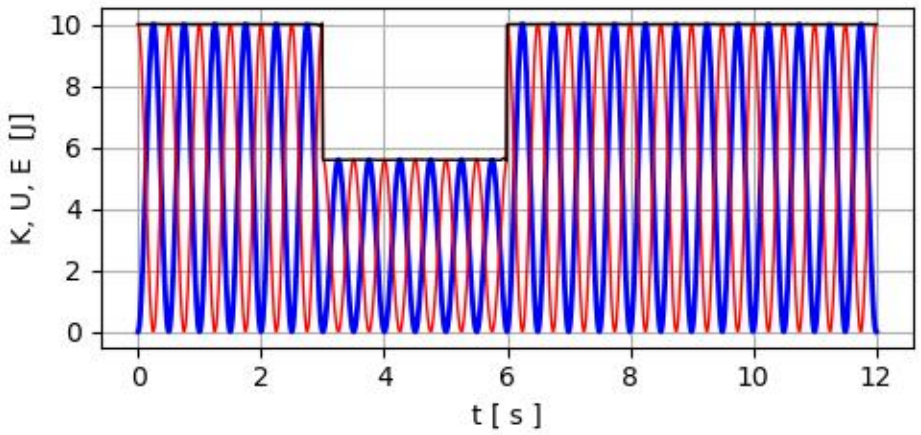



Fig. 3.5. The total energy is a conserved quantity. When the impulsive force acts, work is done by the system so that the total energy of the system is decreased.

### Simulation 4 Sinusoidal driving force sim001.py

You can explore the response of the system to a sinusoidal driving force. You can change the frequency of the driving force and see immediately the response of the system. The system will vibrate at the frequency of the driving force $f_D$.

The equation of motion is

$$m\frac{d^2x(t)}{dt^2} = -k\,x(t) - b\,\frac{dx(t)}{dt} + A\cos\left(2\pi\, f_D t\right)$$

When the driving frequency is near the natural frequency, large amplitude oscillates result with the maximum amplitude occurring very near the natural frequency. The peak amplitude of the

oscillation is determined by the damping parameter $b$. If there is zero damping, then the amplitude of the oscillation grows and grows with time because energy is continuously added to the system. When there is damping, a steady state situation is achieved in which the amplitude of the oscillation reaches a fixed value. This phenomenon is called **resonance**.

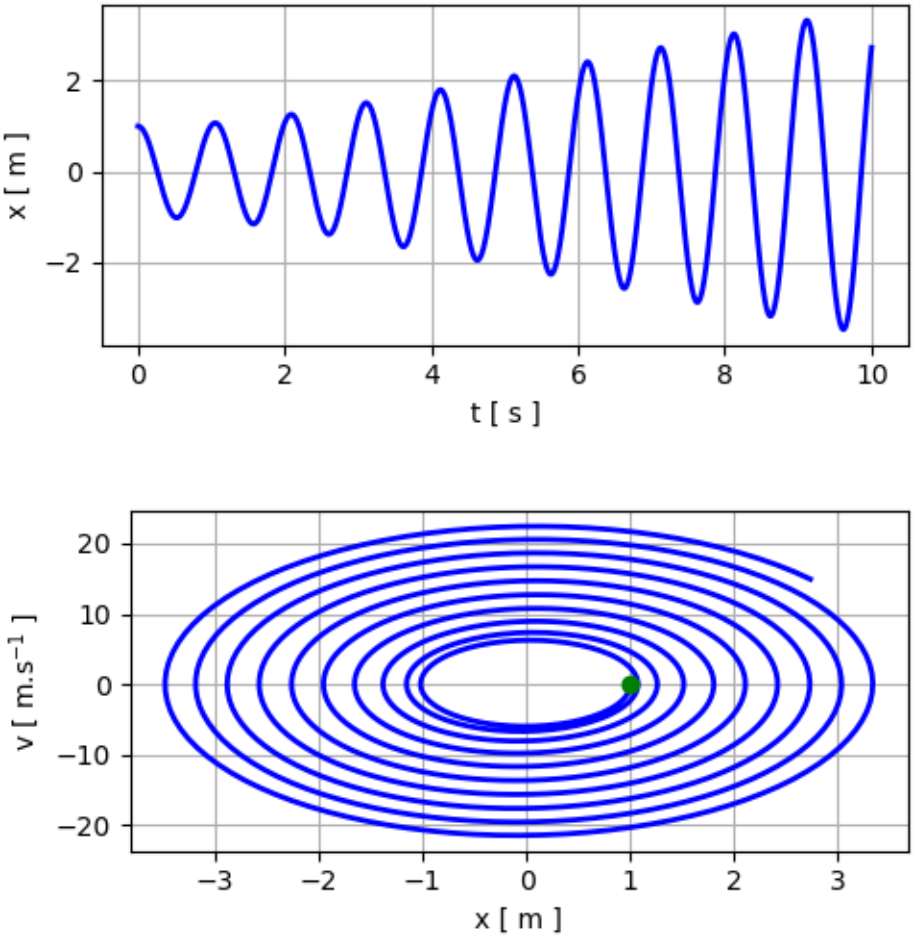



Fig. 4.1. Zero damping $b$ = 0. $f_D$ = 1.02 Hz ($f_0$ = 1.00 Hz). When the system is excited near its natural frequency and there is zero damping, the amplitude of the oscillation increases with time.

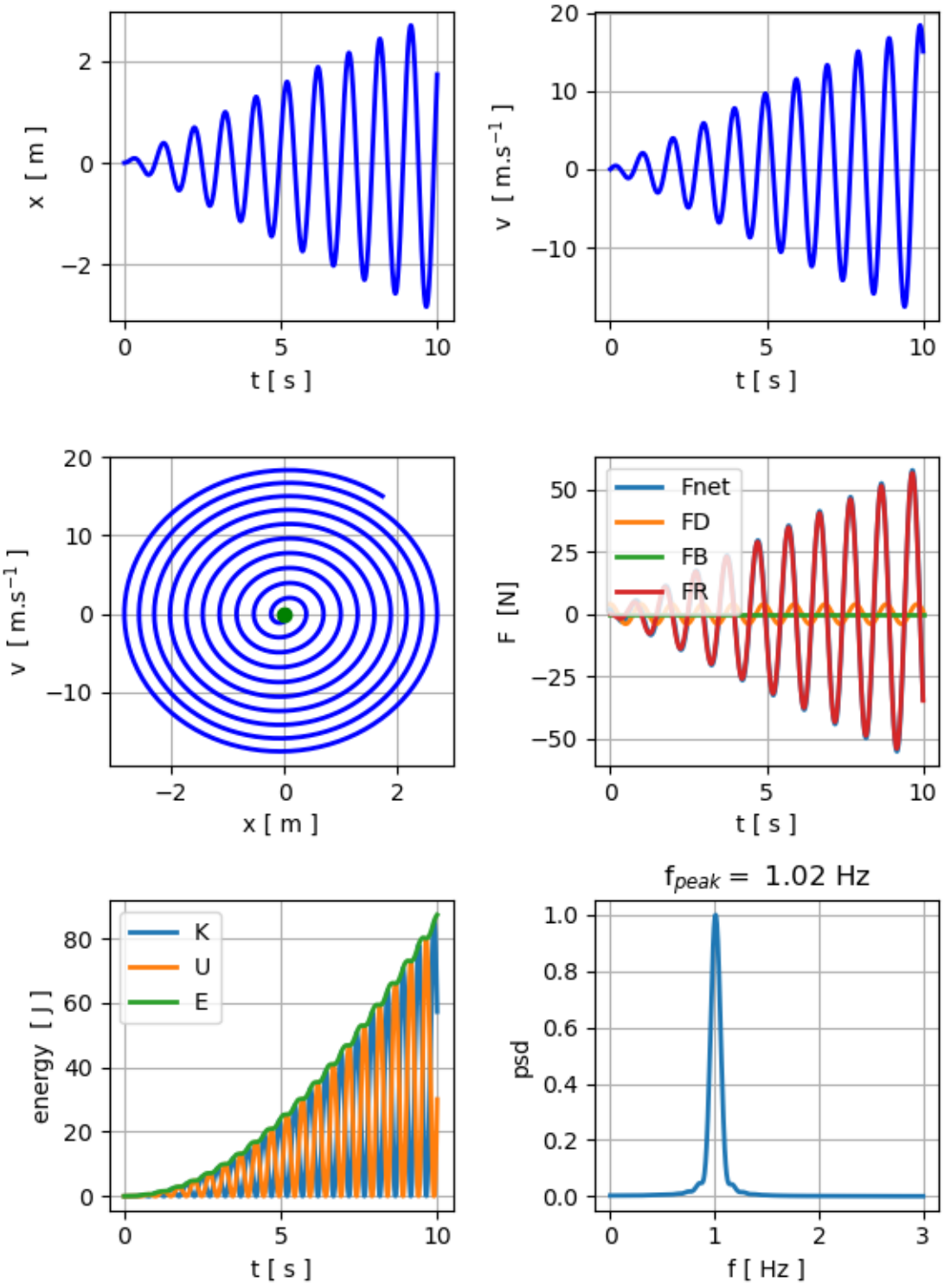



Fig. 4.2. Sinusoidal driving force with zero damping.

When there is damping, then the oscillations will evolve to a fixed amplitude for the oscillation, because the energy input to the system from the external driving force becomes equal to the energy lost from the system because of the damping (figures 4.3, 4.4 ).

Figure 4.3 shows the motion when the driving frequency is close to the natural frequency: $f_0$ = 1.00 Hz, $f_D$ = 1.02 Hz. The resulting amplitude of oscillation is $x_{max}$ = 0.31 m.

Figure 4.4 shows the motion when the driving frequency less than the natural frequency: $f_0$ = 1.00 Hz, $f_D$ = 0.50 Hz, $b$ = 1.0 kg.s$^{-1}$. The resulting amplitude of oscillation is $x_{max}$ = 0.13 m. The system vibrates at the driving frequency and not the natural frequency.

Figure 4.5 shows the motion when the driving frequency greater than the natural frequency: $f_0$ = 1.00 Hz, $f_D$ = 0.350 Hz, $b$ = 1.0 kg.s$^{-1}$. The resulting amplitude of oscillation is small, $x_{max}$ = 0.013 m. The system vibrates dominantly at the driving frequency. The frequency spectrum shows a small peak at the natural frequency (1.00 Hz) and a large peak at the driving frequency (3.5 Hz).

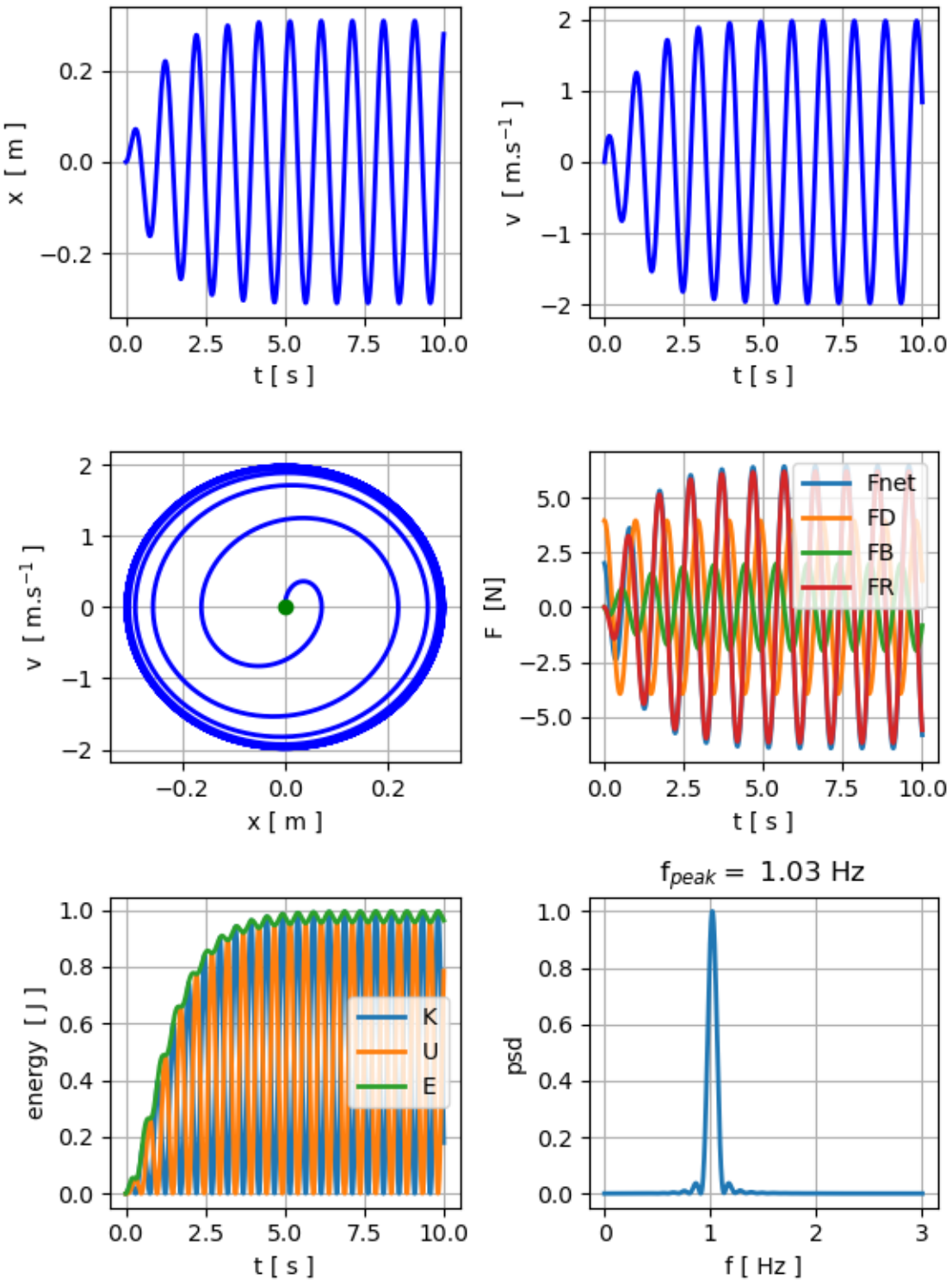



Fig. 4.3. $b$ = 1.0 kg.s$^{-1}$, $f_D$ = 1.02 Hz, $A$ = 2.0 N. Amplitude of oscillation $x_{max}$ = 0.31 m.

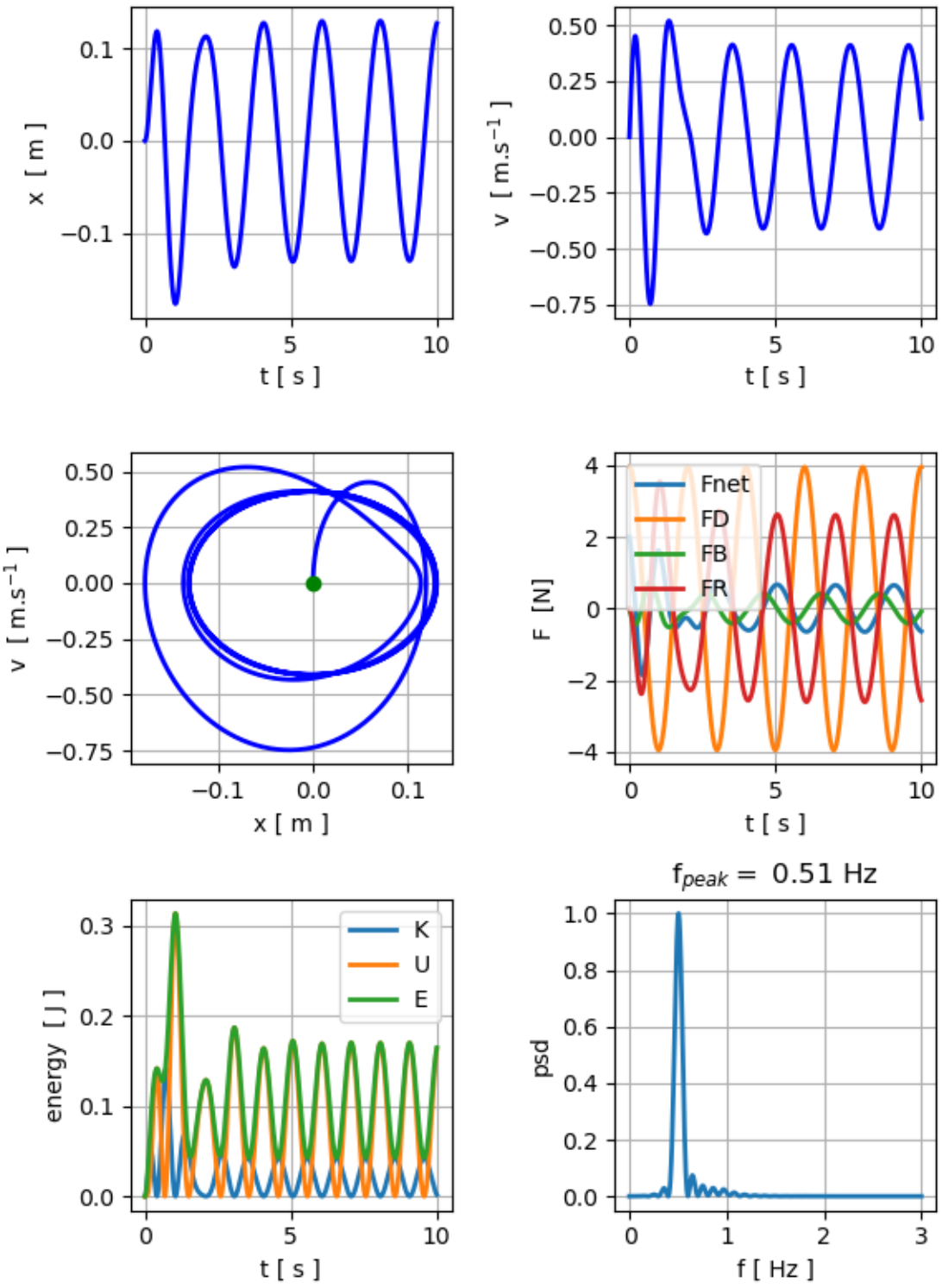



Fig. 4.4. $b$ = 1.0 kg.s$^{-1}$, $f_D$ = 0.50 Hz, $A$ = 2.0 N. Amplitude of oscillation $x_{max}$ = 0.13 m.

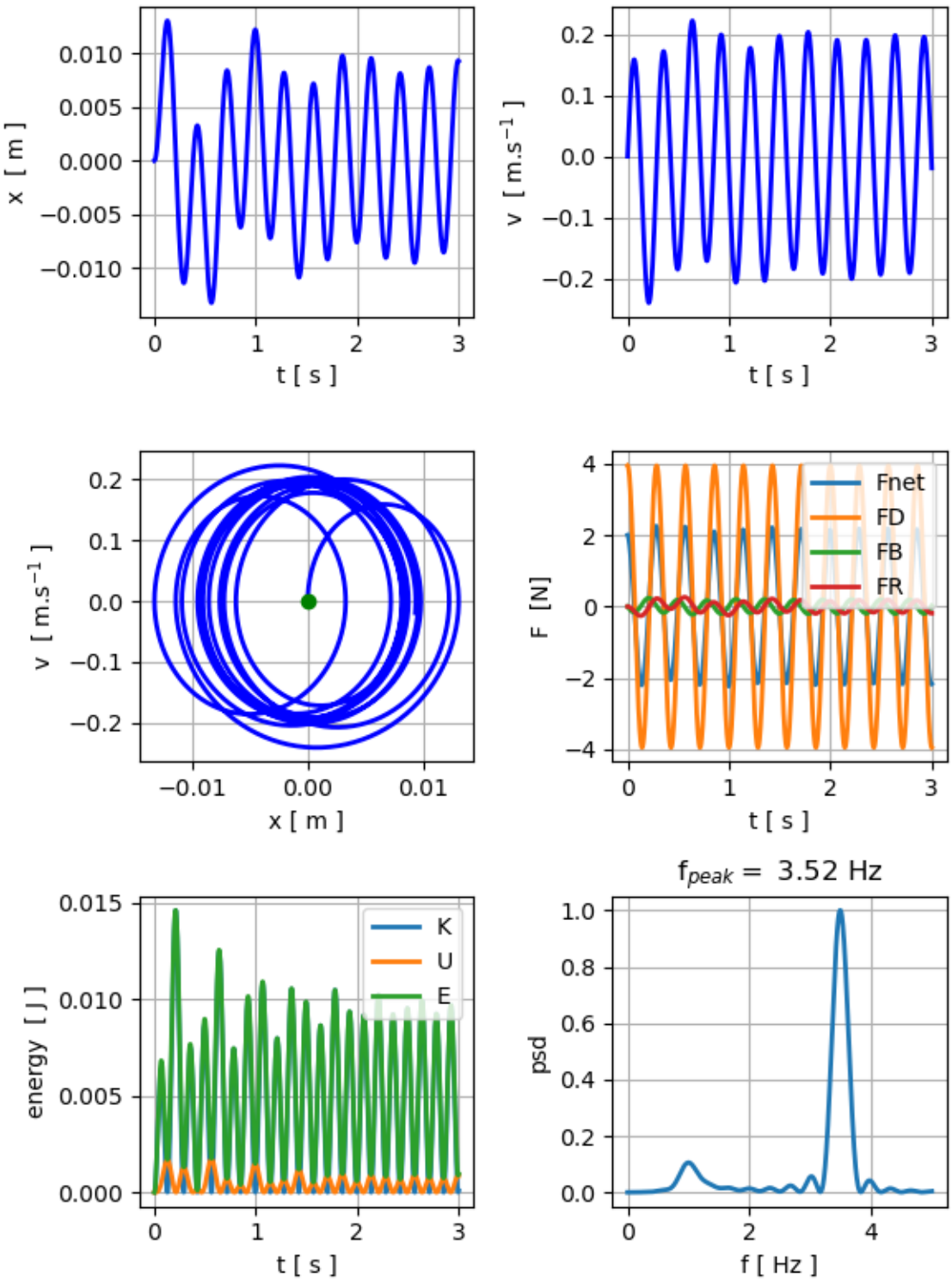



Fig. 4.5. $b$ = 1.0 kg.s$^{-1}$, $f_D$ = 3.50 Hz, $A$ = 2.0 N. Amplitude of oscillation $x_{max}$ = 0.013 m.

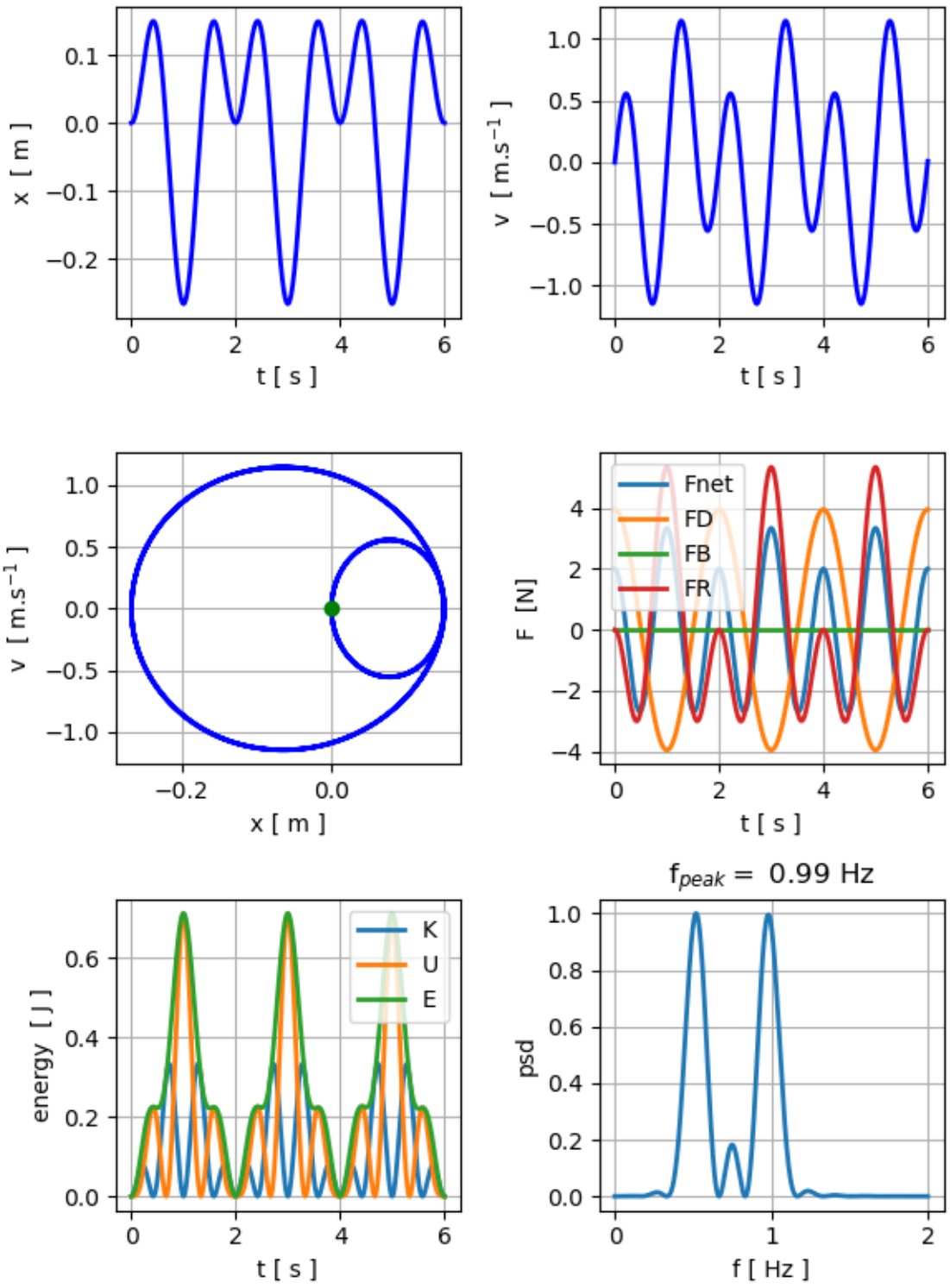



Fig. 4.6. $b$ = 0 kg.s$^{-1}$, $f_D$ = **0.50 Hz**, $A$ = 2.0 N. Superposition of oscillations at frequencies of the natural frequency (1.00 Hz) and the driving frequency (0.50 Hz).

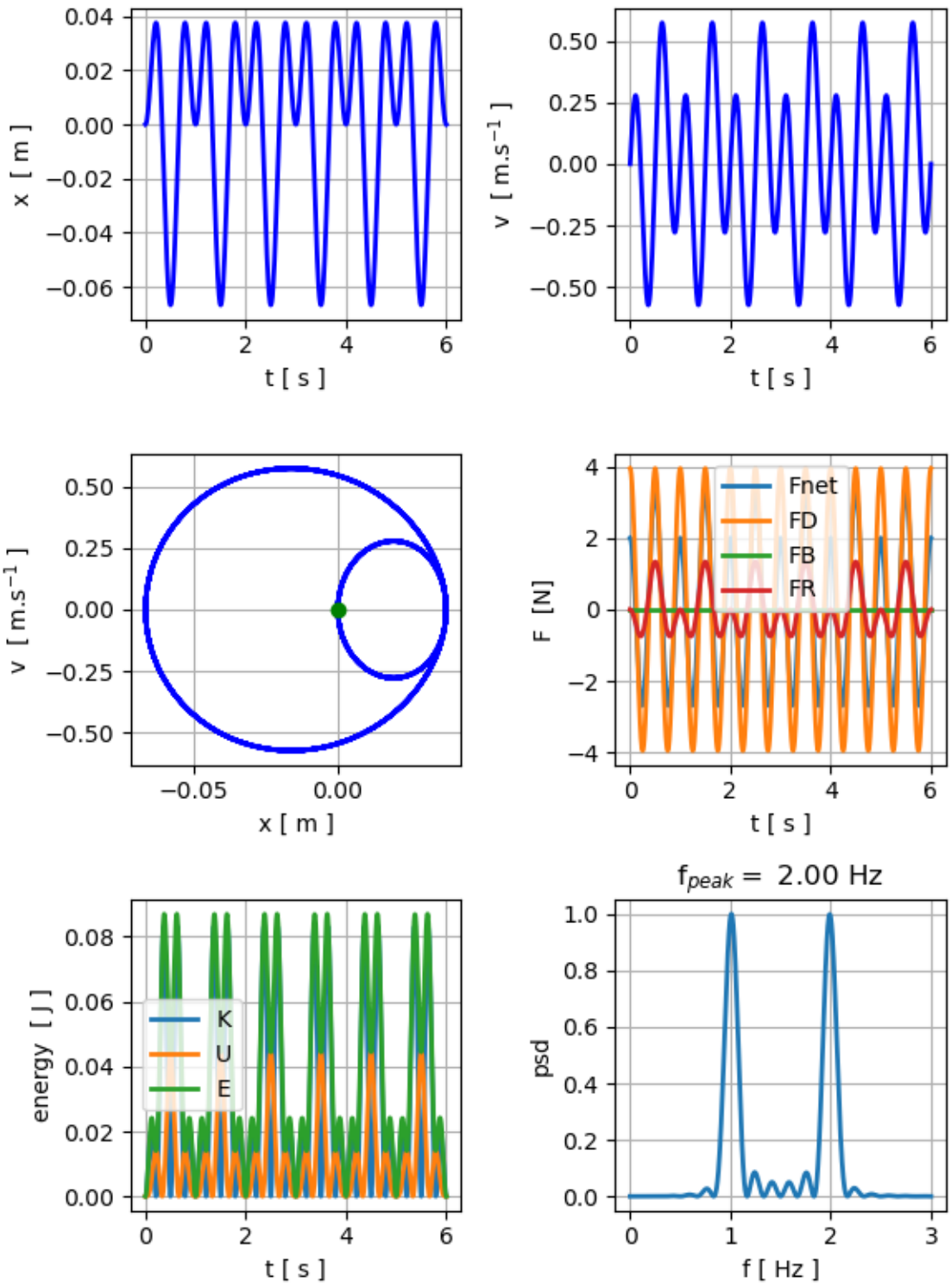



Fig. 4.6. $b$ = 0 kg.s$^{-1}$, $f_D$ = **2.00 Hz**, $A$ = 2.0 N. Superposition of oscillations at frequencies of the natural frequency (1.00 Hz) and the driving frequency (2.00 Hz).

## Simulation 5 Mass -spring response curve sim001B.py

The response curve for the displacemet amplitude of oscillations for the mass - spring system is given by the equation 7

$$(7) \qquad A = \frac{1}{\sqrt{\left(k - m\omega_D{}^2\right)^2 + \left(b\,\omega_D\right)^2}} \qquad \omega_D = 2\pi f_D$$

If the damping coefficient is aero, $b$ = 0, then the amplitude of oscillation approaches infinity as the driving frequency approaches the natural frequency

$$f_D \to f_0 \;\Rightarrow\; A \to \infty$$

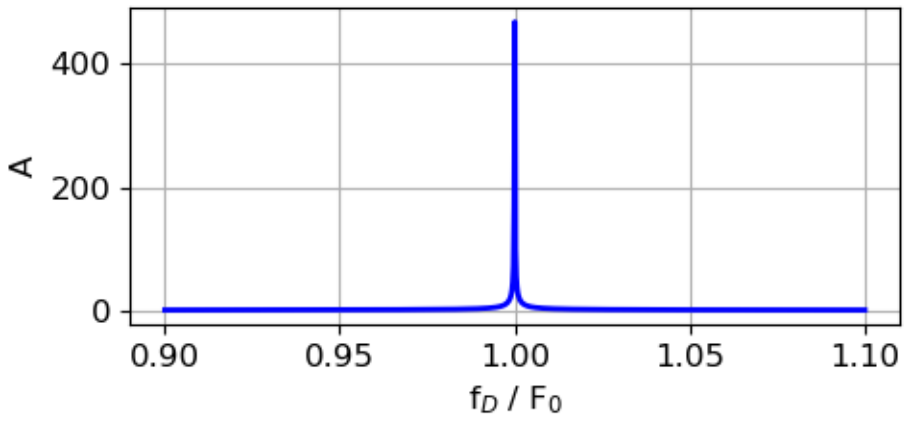



Fig. 5.1. ***b* = 0** $f_0$ = 1.00 Hz $f_D \to f_0 \;\Rightarrow\; A \to \infty$

When the damping is non-zero, then the amplitude of oscillation is restricted to some finite value.

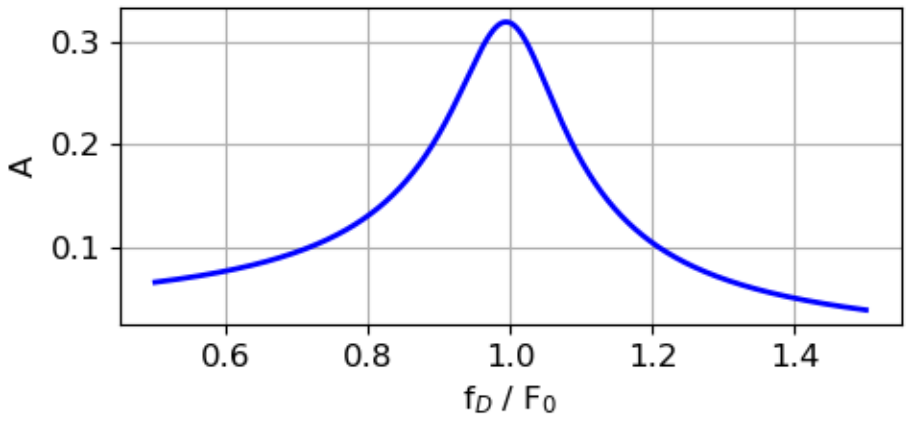



Fig. 5.2. ***b* = 0.50**

$f_0$ = 1.00 Hz $\left(f_D / f_0\right)_{peak} = 0.99$, $A_{max} = 0.32$

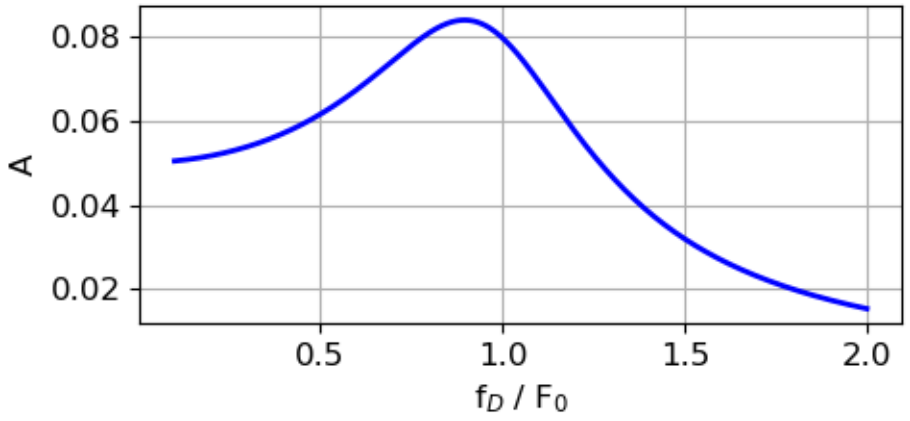



Fig. 5.2. ***b* = 2.0**

$f_0$ = 1.00 Hz $\left(f_D / f_0\right)_{peak} = 0.90$, $A_{max} = 0.084$

When the system is damped, the peak amplitude does not occur at the natural frequency of vibration: $\left(f_D / f_0\right)_{peak} < 1$.

- **Low driving frequency** ($\omega \ll \omega_0$): mass follows the force almost exactly.
- **Near resonance** ($\omega \approx \omega_0$): large amplitude oscillations occur.
- **High driving frequency** ($\omega \gg \omega_n$): mass cannot respond quickly enough, resulting in small amplitudes.

The mass - spring system is fundamental in vibration analysis of vehicle suspensions, earthquake engineering, electrical RLC circuits, and many control systems.