DOING PHYSICS WITH PYTHON

FOURIER METHODS

FOURIER TRANSFORMS

GAUSSIAN FUNCTION

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simFM002.

The Code is used to calculate the Fourier transform, the power spectral density and the inverse Fourier transform functions by the direct integration of the Fourier integrals using Simpson's rule.

```
#%% TIME DOMAIN
Nt = 1999
t1 = -10; t2 = 10
t = linspace(t1,t2,Nt)
tm = 2    # mean time
s = 1     # width of gausssian function
```

```
h = exp(-0.5*(t-tm)**2/s**2)    # Gausssian function

# Expectation values
area = simpson(h,t)          # area
h = h/area                   # normalized Gaussian function
areaT = simpson(h,t)         # check normalization
g = t*h                      # expectation value t
t_avg = simpson(g,t)
g = (t-t_avg)**2*h           # uncertainty in t
var = simpson(g,t)
dt = sqrt(var)


#%%  FREQUENCY DOMAIN
Nf = 1999
f1 = -1; f2 = 1
f = linspace(f1,f2,Nf)
p  = 1j*2*pi
HR = zeros(Nf); HI = HR; H = HR+HI*1j
# Fourier transform
for c in range(Nf):
    g = h*exp(p*f[c]*t)
    H[c] = simpson(real(g),t) + 1j*simpson(imag(g),t)
# Normalize Fourier transform
area = simpson(abs(H),f)
H = abs(H)/area
psd = 2*H**2              # one-sided power spectral density
psd = real(psd/max(psd))    # normalized psd

# Expectation values
g = f*H
f_avg = simpson(g,f)         # expectation value: mean frequency
g = (f - f_avg)**2*abs(H)
```

```
var = simpson(g,f)          # expectation value: uncertainty in f
df = sqrt(var)
dw = 2*pi*df

#%% INVERSE FOURIER TRANSFORM
Hinv = zeros(Nt)
HRinv = zeros(Nf); HIinv = HRinv; Hinv = HRinv + HIinv*1j
for c in range(Nt):
    g = H*exp(-p*f*t[c])
    Hinv[c] = simpson(real(g),f) + 1j*simpson(imag(g),f)
# Normalize Fourier transform

area = simpson(abs(Hinv),t)
Hinv = abs(Hinv)/area
```

INTRODUCTION

Fourier transform methods (or spectral methods) are used in a very large class of computational problems. Many physical processes can be described in the **time domain** by the values of a function $h(t)$ or else in the **frequency domain** $H(f)$. The function $H(f)$ is usually complex function specified by its amplitude and phase. It is useful to think of $h(t)$ and $H(f)$ as being two different representations of the same function. One goes back and forth between these two representations by means of the **Fourier transform equations**

$$(1) \qquad H(f) = \int_{-\infty}^{\infty} h(t) e^{i(2\pi f t)}\, dt$$

$$(2) \qquad h(t) = \int_{-\infty}^{\infty} H(f) e^{-i(2\pi f t)}\, df$$

Equation 1 is the **Fourier transform** and equation 2 gives the **inverse Fourier transform**. If t is measured in seconds, then the frequency f is measured in hertz and the angular frequency ω in radians.

The total power in a signal is the same whether we compute it in the time domain or in the frequency domain. This result is called **Parseval's theorem**

$$(3) \qquad Total\ Power = \int_{-\infty}^{\infty} |h(t)|^2\, dt = \int_{-\infty}^{\infty} |H(f)|^2\, df$$

If one wants to compute how much power there is the frequency interval between f and df, one does not usually distinguish between negative and positive values of f, but rather regards f as varying from zero frequency or DC $(f = 0)$ to $f \to \infty$. In such cases, one defines the **one-sided power spectral density** PSD of the function $h(t)$ as

$$(4) \qquad psd(f) = |H(f)|^2 + |H(-f)|^2$$

If $h(t)$ is a real function, then

$$(5) \qquad psd(f) = 2|H(f)|^2 \qquad 0 \le f < \infty$$

$$(6) \qquad Total\ Power = 2\int_{0}^{\infty} |H(f)|^2\, df$$

Usually, the Fourier transforms given by equations 1 and 2 are calculated by the **fast Fourier transform (FFT)** method. Calculating the Fourier transform by the FFT method has a number of

complications, such as, the sampling rate and frequency domain are **not** independent. Historically, the fast Fourier transform is used because of the speed of the calculations is much faster than the direct evaluation of the Fourier integrals. But, with the speed and memory of modern computers and using software such as Python or Matlab, the computation of the Fourier integrals can be computed by the direct integration without any problems, thus, calculating a **Fourier transform can be made simple**.

All integrals are computed using the Python function **simpson()**.

The time and frequency domains are specified by the variables for the time interval and number of samples:

time domain	t1, t2, Nt
frequency domain	f1, f2, Nf

The frequency domain includes both negative and positive frequencies in computing the inverse Fourier transform (equation 2) and the total power (equation 6). The function $h(t)$ has only non-zero values in the interval time interval from t1 to t2

$$t \le t1 \quad h(t) = 0 \qquad t \ge t2 \quad h(t) = 0$$

The **normalized Gaussian** function $h(t)$ in time t is

(7A) $$h(t) = A_t \exp\left(-\frac{1}{2} \frac{(t - t_m)^2}{\sigma^2} \right)$$

(7B) $$\int_{-\infty}^{\infty} h(t)\, dt = \int_{-\infty}^{\infty} A_t \exp\left(-\frac{1}{2} \frac{(t - t_m)^2}{\sigma^2} \right) dt = 1$$

A_t is the normalization constant (found numerically using Simpson's rule)

t_m mean time (time at which the peak occurs)

σ standard deviation (width of the Gaussian function)

- The **standard deviation** σ of a distribution is a statistical measure that tells you how much, on average, the values in a distribution deviate (spread out) from its mean value (determines how wide or narrow the peak is). The standard deviation is the square root of the variance, which is the average of the squared differences from the mean.

Since the Gaussian function given by equation 1 is normalized, then the mean time and standard deviation can be computed as expectation values

(8A) $$t_m = < t > = \int_{-\infty}^{\infty} t\, h(t)\, dt$$

(8B) $\sigma^2 = \left\langle \left(t - t_m\right)^2 \right\rangle = \int_{-\infty}^{\infty} \left(t - t_m\right)^2 h(t)\, dt$ $\sigma_t^{\,2}$ variance

The standard deviation σ_t represents an uncertainty in time (dt)

The Fourier transform function is also normalized

(9) $A_f \int_{-\infty}^{\infty} H(f)\, df = 1$ A_f normalization constant

$A_f\, H(f) \rightarrow H(f)$

The uncertainty in the frequency is computed from the expectation values

(10A) $f_m = < f > = \int_{-\infty}^{\infty} f\, H(f)\, df$

(10B) $\sigma_f^{\,2} = \left\langle \left(f - f_m\right)^2 \right\rangle = \int_{-\infty}^{\infty} \left(f - f_m\right)^2 H(f)\, df$

The standard deviation σ_f represents an uncertainty in frequency (df)

The Fourier transform of a Gaussian function in time t is Gaussian function in frequency $f\ (\omega)$. Thus, the Fourier transform, apart from scaling factors, does not change the shape of the function. No other nontrivial function has this perfect balance between localization in both domains, which is why Gaussian pulses, Gaussian beams, and Gaussian wave packets appear throughout signal processing, optics, and quantum mechanics.

A beautiful way to remember it is:

The Gaussian is the "most localized" function simultaneously in a domain and its Fourier domain, and therefore saturates the uncertainty bound.

The Fourier transform of a Gaussian is another Gaussian. The narrower the distribution in time the broader the distribution in frequency. The uncertainties (widths) $\left(\Delta t = \sqrt{\langle t - t_m \rangle^2} \right)$ and $\left(\Delta\omega = \sqrt{\langle \omega - \omega_m \rangle^2} \right)$ are roughly inverse to each other such that

$$\Delta t\,\Delta\omega \geq \tfrac{1}{2} \quad \Delta t\,\Delta f \geq \tfrac{1}{4\pi} \qquad \omega = 2\pi f$$

The Gaussian function is unique because it achieves the absolute minimum possible product of widths in the time domain and frequency domain.

- σ_t small: A sharp, narrow spike localized tightly in time.
- σ_t large: A broad, flat curve spread out in time

SIMULATION **Gaussian function** **simFM002.py**

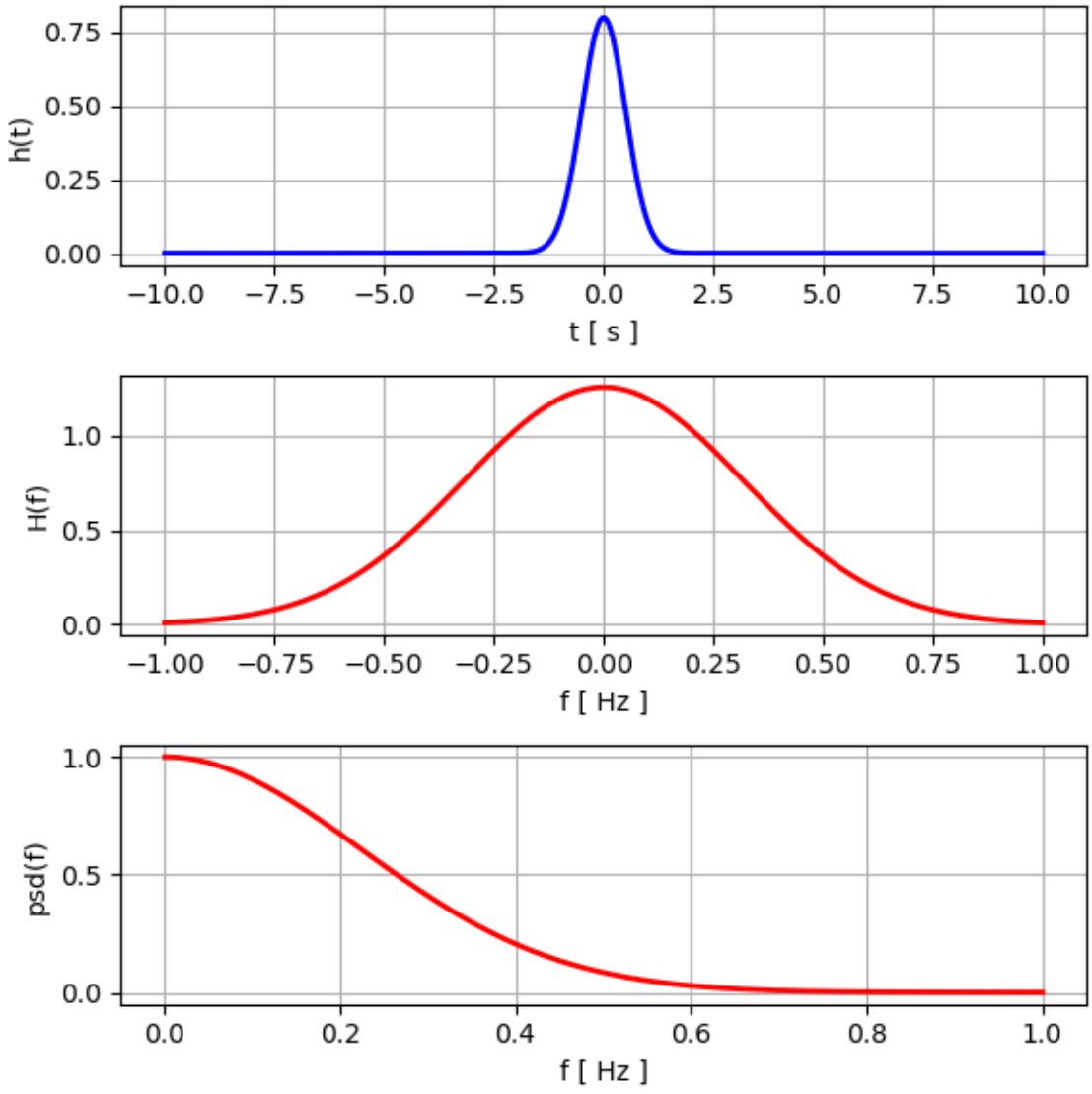


Fig. 1. A plot of a normalized Gaussian function, Fourier transform and the one-sided power spectral density.

tm = 0 s, σ_t = 0.50 s.

Expectation values : dt = 0.50 s, df = 0.32 Hz,

Uncertainty principle: dt df = 0.16 > 0.08

Uncertainty principle: dt dw = 0.99 > 0.5

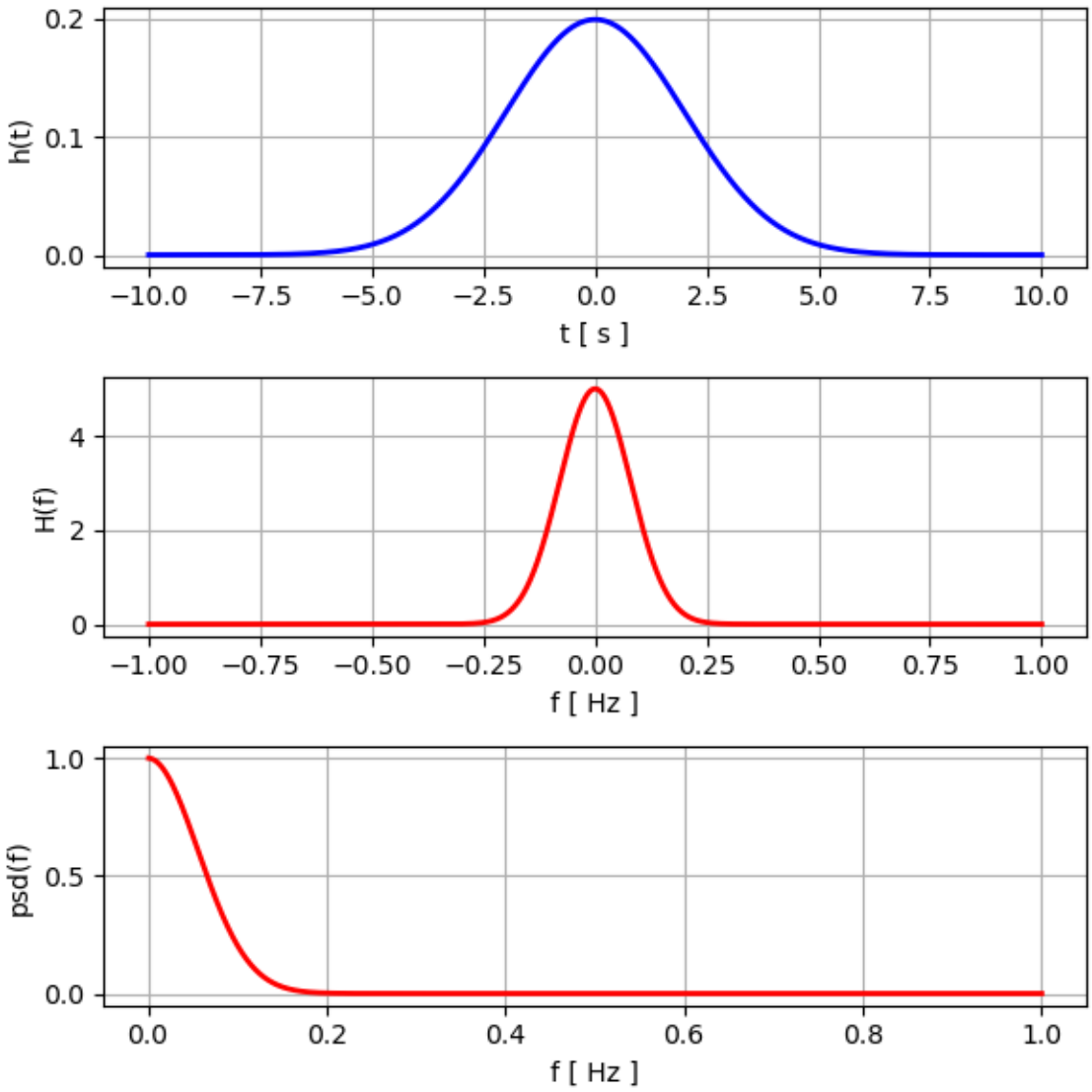


Fig. 2. A plot of a normalized Gaussian function, Fourier transform and the one-sided power spectral density.

tm = 0 s, σ_t = 2.00 s.

Expectation values : dt = 02.00 s, df = 0.080 Hz,

Uncertainty principle: dt df = 0.16 > 0.08

Uncertainty principle: dt dw = 1.00 > 0.5

A narrow Gaussian signal has a wide spectrum whereas a wide Gaussian signal has a narrow spectrum such that $\Delta t \, \Delta\omega \geq 1/2$.

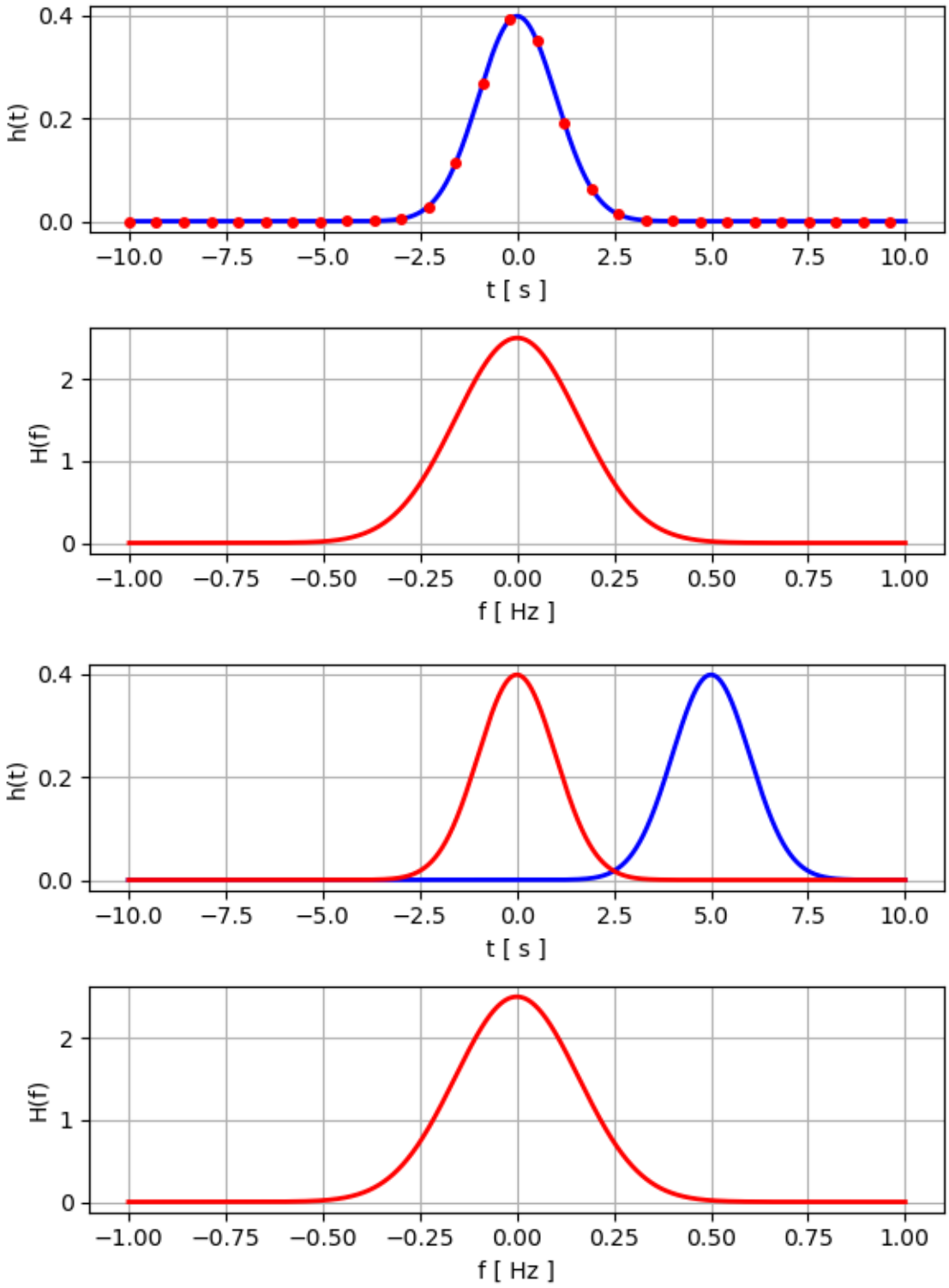


Fig. 3. Gaussian function, Fourier transform and the inverse Fourier transform: σ_t = 1.00 s, tm = 0 (top) and tm = 5.0 s (bottom).